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REASONING ABOUT TEMPORAL CONSTRAINTS: CLASSIFYING THE COMPLEXITY IN ALLEN'S ALGEBRA BY USING AN ALGEBRAIC TECHNIQUE

Andrei Krokhin, Peter Jeavons
Computing Laboratory
University of Oxford, UK
e-mail: {Andrei.Krokhin, Peter.Jeavons}@comlab.ox.ac.uk

Peter Jonsson
Department of Computer and Information Science
Linköping University, Sweden
e-mail: petej@ida.liu.se

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Oxford University Computing Laboratory
Wolfson Building, Parks Road, Oxford OX1 3QD

Abstract

Allen's interval algebra is one of the best established formalisms for temporal reasoning. We study those fragments of Allen's algebra that contain a basic relation distinct from the equality relation. We prove that such a fragment is either NP-complete or else contained in some already known tractable subalgebra. We obtain this result by giving a new uniform description of known maximal tractable subalgebras and then systematically using an algebraic technique for description of maximal subalgebras with a given property. This approach avoids the need for extensive computer-assisted search.

1 Introduction

Reasoning about temporal constraints is an important task in many areas of computer science and elsewhere, such as scheduling [26], natural language processing [28], planning [2], technical diagnosis [24], and molecular biology [6]. Several frameworks for formalizing this type of problems have been suggested; for instance, the *point algebra* [31] (for expressing relations between time points), the *point-interval algebra* [33] (for expressing relations between time points and intervals) and the famous *Allen's algebra* [1] for expressing relations between time intervals. This algebra has proven to be useful and convenient and it has also become the kernel of some other formalisms [3, 21] .

Due to computational hardness (the basic satisfiability problem in Allen's algebra is NP-complete [34]), it is unlikely that efficient algorithms exists for reasoning in the full algebra. The computational difficulty has motivated the search for effective heuristics, e.g. [17, 22, 32], and the study of the complexity of fragments of the algebra, e.g., [7, 8, 11, 23]. In this paper we follow the second approach assuming that $P \neq NP$. Research in this direction has focused on identifying *maximal* tractable fragments, i.e., fragments which cannot be extended without losing tractability. So far, eighteen maximal tractable fragments of the algebra have been identified [7, 8, 23]. It is still unknown whether any others exist.

A complete classification of complexity within a certain large part of Allen's algebra was obtained in [9]. This result (as well as most similar results) was achieved by computer-assisted exhaustive search. However, it was noted in [9] that, for further progress, theoretical studies of the structure of Allen's algebra are required. There are some theoretical investigations of the structure of Allen's algebra, (see, e.g., [12, 13, 16]). However they generally allow more operations on relations than originally used in [1], which makes them inappropriate for classifying complexity within the interval algebra. In this paper we systematically use algebraic method that is similar to the approach taken in [18].

The first novel element in our approach is a new uniform description for all of the maximal tractable subalgebras of Allen's algebra which have already been identified (Table 3). Then we exhibit several new small NP-complete fragments of the algebra, including the first examples of NP-complete singletons (Proposition 2). Finally, we fully exploit the algebraic properties of Allen's algebra by importing a technique from general algebra. This technique has been used in many other contexts to obtain a description of maximal subalgebras of a given algebra with a given property (e.g., [29, 37]). Here, for the first

Basic relation		Example	Endpoints
x precedes y	p	xxx yyy	$x^+ < y^-$
y preceded by x	p^{-1}		
x meets y	m	xxxx yyyy	$x^+ = y^-$
y met-by x	m^{-1}		
x overlaps y	o	xxxx yyyy	$x^- < y^- < x^+$, $x^+ < y^+$
y overl.-by x	o^{-1}		
x during y	d	xxx yyyyyyy	$x^- > y^-$, $x^+ < y^+$
y includes x	d^{-1}		
x starts y	s	xxx yyyyyyy	$x^- = y^-$, $x^+ < y^+$
y started by x	s^{-1}		
x finishes y	f	xxx yyyyyyy	$x^+ = y^+$, $x^- > y^-$
y finished by x	f^{-1}		
x equals y	$\equiv$	xxxx yyyy	$x^- = y^-$, $x^+ = y^+$

Table 1: The thirteen basic relations. The endpoint relations $x^- < x^+$ and $y^- < y^+$ that are valid for all relations have been omitted.

time, we systematically apply this technique to Allen's algebra to obtain a complete classification of the complexity of those fragments of the algebra that contain a non-trivial basic relation (Theorem 1). Our method also provides a new elementary proof of maximality of ten tractable subalgebras. In previous papers this proof of maximality has required extensive computer-aided case analysis [7, 8].

The paper is organized as follows: in Section 2 we give the basic definitions of Allen's algebra and present the known maximal tractable subalgebras in the new form. In Section 3 we discuss an algebraic technique used for obtaining our results and compare it with the computer-aided method used for the similar purpose in [9]. Section 4 contains the new classification result and its proof, and Section 5 contains a brief discussion of future extensions.

2 Allen's Algebra

Allen's interval algebra [1] is based on the notion of *relations between intervals*. An interval x is represented as a tuple (x^-, x^+) of real numbers with $x^- < x^+$, denoting the left and right endpoints of the interval, respectively. The possible relations between intervals are the 2^{13} possible unions of 13 *basic interval relations*[1], which are shown in Table 1. For the sake of brevity, relations between intervals will be written as collections of basic relations. So, for instance, the union of the basic intervals p, m and f^{-1} is written (pmf^{-1}). Allen's algebra $\mathcal{A}$ consists of the 2^{13} possible relations between intervals together

[1]Note that the basic relations are jointly exhausitive and pairwise disjoint in the sense that any two given intervals are related by exactly one basic relation.

with the operations *converse* $\cdot^{-1}$, *intersection* $\cap$ *and composition* $\circ$ which are defined as follows:

$$\forall x, y: \ xr^{-1}y \Leftrightarrow yrx$$

$$\forall x, y: \ x(r \cap s)y \Leftrightarrow xry \ \& \ xsy$$

$$\forall x, y: \ x(r \circ s)y \Leftrightarrow \exists z: \ (xrz \ \& \ zsy)$$

It follows that the converse of $r = (b_1 \dots b_n)$ is equal to $(b_1^{-1} \dots b_n^{-1})$. The intersection of two relations can be expressed as the usual set-theoretic intersection. Since the basic relations are pairwise disjoint, the intersection of two relations $r_1, r_2 \in \mathcal{A}$ consists of the basic relations that are present in both r_1 and r_2. Using the definition of composition, it can be derived that

$$(b_1 \dots b_n) \circ (b'_1 \dots b'_m) = \bigcup \{b_i \circ b'_j \mid i \leq n, \ j \leq m\}.$$

Table 2 essentially simplifies the checking of algebraic calculations that are given in Section 4.

The problem of *satisfiability* ($\mathcal{A}$-SAT) of a set of interval variables with relations between them is that of deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals are satisfied. This is defined as follows.

Definition 1 *Let $X \subseteq \mathcal{A}$ be a set of interval relations. An instance I of $\mathcal{A}$-SAT(X) is a set, V, of variables and a set of constraints of the form xry where $x, y \in V$ and $r \in X$. The question is whether I is* satisfiable*, i.e., whether there exists a function, f, from V to the set of all intervals such that $f(x)\, r\, f(y)$ holds for every constraint xry in I. Any such function f is called a* model *of I.*

If there exists a polynomial-time algorithm solving all instances of $\mathcal{A}$-SAT(X) then we say that X is tractable. On the other hand, if $\mathcal{A}$-SAT(X) is NP-complete then we say that X is NP-complete. Since the problem $\mathcal{A}$-SAT$(\mathcal{A})$ is NP-complete [34], there arises the question of description of tractable subsets of Allen's algebra.

Subsets of $\mathcal{A}$ that are closed under the operations of intersection, converse and composition are said to be *subalgebras*. For a given subset X of $\mathcal{A}$, the smallest subalgebra containing X is called the subalgebra *generated* by X and is denoted by $\langle X \rangle$. It is easy to see that $\langle X \rangle$ is obtained from X by adding all relations that can be obtained from the relations in X by using the three operations of $\mathcal{A}$.

It is known [23], and easy to prove, that, for every $X \subseteq \mathcal{A}$, the problem $\mathcal{A}$-SAT$(\langle X \rangle)$ is polynomially equivalent to $\mathcal{A}$-SAT(X). Therefore, to classify the complexity of all subsets of $\mathcal{A}$ it is only necessary to consider *subalgebras* of $\mathcal{A}$. Obviously, adding relations to a subalgebra can only increase the complexity of the corresponding satisfiability problem. Thus, since $\mathcal{A}$ is finite, the problem of describing tractability in $\mathcal{A}$ can be reduced to the problem of describing the *maximal* tractable subalgebras in $\mathcal{A}$, i.e., subalgebras that cannot be extended without losing tractability.

$\circ$	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
$\equiv$	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}	d	d^{-1}	s	s^{-1}	f	f^{-1}
p	p	p	$\top$	p	ρ	p	ρ	ρ	p	p	p	ρ	p
p^{-1}	p^{-1}	$\top$	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	λ^{-1}	p^{-1}	p^{-1}	p^{-1}
m	m	p	ρ^{-1}	p	θ	p	β	β	p	m	m	β	p
m^{-1}	m^{-1}	λ	p^{-1}	σ	p^{-1}	γ^{-1}	p^{-1}	γ^{-1}	p^{-1}	γ^{-1}	p^{-1}	m^{-1}	m^{-1}
o	o	p	ρ^{-1}	p	β^{-1}	α	ν	β	λ	o	γ	β	α
o^{-1}	o^{-1}	λ	p^{-1}	γ	p^{-1}	ν	α^{-1}	γ^{-1}	ρ^{-1}	γ^{-1}	α^{-1}	o^{-1}	β^{-1}
d	d	p	p^{-1}	p	p^{-1}	ρ	λ^{-1}	d	$\top$	d	λ^{-1}	d	ρ
d^{-1}	d^{-1}	λ	ρ^{-1}	γ	β^{-1}	γ	β^{-1}	ν	d^{-1}	γ	d^{-1}	β^{-1}	d^{-1}
s	s	p	p^{-1}	p	m^{-1}	α	γ^{-1}	d	λ	s	σ	d	α
s^{-1}	s^{-1}	λ	p^{-1}	γ	m^{-1}	γ	o^{-1}	γ^{-1}	d^{-1}	σ	s^{-1}	o^{-1}	d^{-1}
f	f	p	p^{-1}	m	p^{-1}	β	α^{-1}	d	ρ^{-1}	d	α^{-1}	f	θ
f^{-1}	f^{-1}	p	ρ^{-1}	m	β^{-1}	o	β^{-1}	β	d^{-1}	o	d^{-1}	θ	f^{-1}

$\alpha = (\mathsf{pmo}) \quad \beta = (\mathsf{ods}) \quad \gamma = (\mathsf{od}^{-1}\mathsf{f}^{-1}) \quad \sigma = (\equiv \mathsf{ss}^{-1}) \quad \theta = (\equiv \mathsf{ff}^{-1})$

$\rho = (\mathsf{pmods}) \quad \lambda = (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \quad \nu = (\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$

$\top = (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$

Table 2: Composition table for the basic relations in Allen's algebra

The known maximal tractable subalgebras [7, 8, 23] are presented in Table 3. In our proofs as well as in Table 3 we use the symbol $\pm$, which should be interpreted as follows. A condition involving $\pm$ means the conjunction of two conditions: one corresponding to $+$ and one corresponding to $-$. For example, condition $(\mathsf{o})^{\pm 1} \subseteq r \Leftrightarrow (\mathsf{d})^{\pm 1} \subseteq r$ means that both $(\mathsf{o}) \subseteq r \Leftrightarrow (\mathsf{d}) \subseteq r$ and $(\mathsf{o}^{-1}) \subseteq r \Leftrightarrow (\mathsf{d}^{-1}) \subseteq r$ hold. The main advantage of using the $\pm$ symbol is conciseness: in any subalgebra of $\mathcal{A}$, the $+$ and the $-$ conditions are satisfied (or not satisfied) simultaneously, and, therefore only one of them needs to be verified.

In previous papers, the subalgebras from Table 3 were defined in other ways. However, in all cases except for $\mathcal{H}$, it is very straightforward to verify that our definitions are equivalent to the original ones. The subalgebra $\mathcal{H}$ was originally defined as the 'ORD-Horn algebra' [23], but has also been characterized as the algebra of 'pre-convex' relations (see, e.g., [18]). Using the latter description it is not hard to show that our definition of $\mathcal{H}$ is equivalent. Finally, we note that there is one more known maximal tractable subalgebra that is not included in Table 3. That subalgebra is defined as $\{\emptyset\} \cup \{r \in \mathcal{A} \mid (\equiv) \subseteq r\}$ and the only basic relation it contains is $(\equiv)$.

3 The Algebraic Background

In general algebra, there often arises the problem of identifying those substructures of a given structure[2] that have a property ϕ satisfying the following condition: if some substructure possesses ϕ then so does any substructure contained in it. Note that tractability of a subalgebra is an example of such a property in Allen's algebra.

A solution to the above problem is often represented as the list of all *maximal* substructures possessing ϕ, that is substructures that possess ϕ and cannot be extended without losing this property. Such maximal substructures may not exist in infinite structures, but in finite ones (such as Allen's algebra) they always do.

Among such properties ϕ, the property of being proper (that is, distinct from the whole structure) has been studied most frequently [5, 19, 27, 30, 36, 39]; maximal proper substructures are called simply maximal. As a classic example, Sylow subgroups that are, as is well known in algebra, maximal p-subgroups play a very important role in group theory (see any textbook on group theory, e.g., [4]). The above problem has also been investigated in different algebraic contexts for many other properties ϕ (see, e.g., [29, 35, 37, 38]).

Now let us describe an approach for obtaining classification results of the kind discussed above. Suppose that we have a structure A, a property ϕ as described above, and a list $S_1, \ldots, S_n$ of pairwise incomparable (with respect to inclusion) substructures of A such that every S_i possesses ϕ. Suppose also that we want to prove that any substructure of A possesses the property ϕ if and only if it is contained in one of $S_1, \ldots, S_n$, that is, that this list consists of all maximal substructures of A that possess ϕ. Then it suffices to check that every subset of A that is not entirely contained in one of the S_i gener-

[2] Most frequently, subalgebras of a given algebra or subvarieties of a given variety; varieties are very important and widely studied classes of algebras (see, e.g., [20] for general introduction).

$$\mathcal{S}_{\mathsf{p}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}$$
$$\mathcal{S}_{\mathsf{d}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d}^{-1})^{\pm 1} \subseteq r\}$$
$$\mathcal{S}_{\mathsf{o}} = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\}$$
$$\mathcal{A}_1 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s}^{-1})^{\pm 1} \subseteq r\}$$
$$\mathcal{A}_2 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$$
$$\mathcal{A}_3 = \{r \mid r \cap (\mathsf{pmodf})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$$
$$\mathcal{A}_4 = \{r \mid r \cap (\mathsf{pmodf}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\}$$

$$\mathcal{E}_{\mathsf{p}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\}$$
$$\mathcal{E}_{\mathsf{d}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r\}$$
$$\mathcal{E}_{\mathsf{o}} = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\}$$
$$\mathcal{B}_1 = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$$
$$\mathcal{B}_2 = \{r \mid r \cap (\mathsf{pmods})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f})^{\pm 1} \subseteq r\}$$
$$\mathcal{B}_3 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{s}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$$
$$\mathcal{B}_4 = \{r \mid r \cap (\mathsf{pmod}^{-1}\mathsf{s})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}$$

$$\mathcal{E}^* = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{pmod})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ff}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{S}^* = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{pmod}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ss}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{H} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{os})^{\pm 1} \neq \emptyset \ \&\ r \cap (\mathsf{o}^{-1}\mathsf{f})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ds})^{\pm 1} \neq \emptyset \ \&\ r \cap (\mathsf{d}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r, \text{ and} \\ 3)\ r \cap (\mathsf{pm})^{\pm 1} \neq \emptyset \ \&\ r \not\subseteq (\mathsf{pm})^{\pm 1} \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r \end{array} \right\}$$

Table 3: Maximal tractable subalgebras of Allen's algebra.

ates a substructure violating ϕ. The checking can be done by exhaustive computer-aided enumeration, as in [14, 15]. However, applicability of this method is very limited, since even small structures require significant amount of time to be handled this way. Besides, this method ignores the considerable similarity of many of the calculations. In view of this, it is generally much more effective to use a more analytical method which breaks down the proof into a collection of simple cases. So one usually chooses a system Σ of conditions (which possibly interact in a sophisticated way) such that every subset of X satisfies at least one of these conditions, and then, for each condition $\sigma \in \Sigma$, one proves the required result assuming that σ is satisfied. Obviously, this provides a proof of the desired classification result. The choice of Σ must make algebraic calculations feasible. It therefore strongly depends on the structure A and the property ϕ, and is usually based on some algebraic intuition. In our proofs, for example, in Section 4.2, the preconditions of Lemmata 5-10 form such a system Σ, and Lemma 3 provides the proof that, in any case, at least one of these precondition holds.

In the study of the complexity of temporal reasoning, such an algebraic approach has been used in [18] to give an elementary proof of the fact that, in Allen's algebra, the ORD-Horn subalgebra is the only maximal tractable subalgebra containing all basic relations. This approach has not previously been applied to the complete analysis of tractability in the whole of Allen's algebra.

4 Main Result

For a subalgebra $\mathcal{S}$ of $\mathcal{A}$, we denote by $bas(\mathcal{S})$ the set of basic relations in $\mathcal{S}$. Our result is extension of the following proposition which was proved in [9].

Proposition 1
1) Let $\mathcal{S}$ be a subalgebra of $\mathcal{A}$ with $|bas(\mathcal{S})| > 3$. Then $\mathcal{S}$ is tractable if and only if it is contained in one of the following 7 algebras: $\mathcal{S}_{\mathsf{p}}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{E}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{d}}$, $\mathcal{E}_{\mathsf{o}}$, and $\mathcal{H}$. Otherwise $\mathcal{S}$ is NP-complete.
2) Let $\mathcal{S}$ be a subalgebra of $\mathcal{A}$ such that $(\mathsf{m}) \in \mathcal{S}$ or $(\mathsf{p}) \in \mathcal{S}$. If $\mathcal{S}$ is tractable then $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{p}}$, or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{p}}$, or $\mathcal{S} \subseteq \mathcal{H}$. Otherwise $\mathcal{S}$ is NP-complete.

We improve this result by showing that any so far unknown tractable subalgebra can contain only a *trivial* basic relation. Throughout the paper, by a *trivial* basic relation we mean the equality relation $\equiv$.

Theorem 1 *Suppose $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ containing a non-trivial basic relation. Then $\mathcal{S}$ is tractable if and only if it is contained in one of the 17 algebras listed in Table 3. Otherwise $\mathcal{S}$ is NP-complete.*

We can without loss of generality assume that $\mathcal{S}$ contains the relation $(\equiv)$, since it is easy to show that $\mathcal{S}$ and $\mathcal{S} \cup \{(\equiv)\}$ have the same complexity (up to polynomial-time equivalence). This implies that the size of $bas(\mathcal{S})$ is odd since $\mathcal{S}$ is closed under converse.

By combining Proposition 1(1) and the previous observation, we may consider only the case $|bas(\mathcal{S})| = 3$. By Proposition 1(2), it suffices to consider the cases where $bas(\mathcal{S})$ is one of the following sets: $\{\equiv, \mathsf{d}, \mathsf{d}^{-1}\}$, $\{\equiv, \mathsf{o}, \mathsf{o}^{-1}\}$, $\{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$, and $\{\equiv, \mathsf{f}, \mathsf{f}^{-1}\}$. We do this in Subsections 4.2, 4.3 and 4.4. Subsection 4.1 contains necessary NP-completeness results.

It can be observed that when $\mathcal{S}$ is one of the algebras $\mathcal{E}^*$, $\mathcal{S}^*$, $\mathcal{A}_i$, or $\mathcal{B}_i$ $(1 \leq i \leq 4)$, then $bas(\mathcal{S})$ is either $\{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$ or $\{\equiv, \mathsf{f}, \mathsf{f}^{-1}\}$. With the help of computer-aided exhaustive search, it was proved in [7, 8] that these ten algebras are maximal tractable subalgebras of Allen's algebra. Since we do not use computer-aided results in Subsection 4.4, our results provide a new elementary proof of maximality for these algebras.

Finally, we give a notation that will constantly be used in our proofs. If b is a basic relation then r_b denotes the least relation $r \in \mathcal{S}$ such that $(b) \subseteq r$, i.e. the intersection of all $r \in \mathcal{S}$ with this property. It is easy to show that the relation r_b exists in $\mathcal{S}$ for every basic relation b, except for two very simple cases: when every $r \in \mathcal{S}$ satisfies $r \subseteq (\equiv \mathsf{ss}^{-1})$ or $r \subseteq (\equiv \mathsf{ff}^{-1})$. In these cases, we have $\mathcal{S} \subseteq \mathcal{H}$. Given a relation r, we write r^* to denote the relation $r \cap r^{-1}$.

Throughout this section we shall assume that a non-trivial relation is also non-empty; this will cause no confusion.

4.1 NP-completeness results

This subsection contains the NP-completeness results that we need in the sequel. Our reductions are based on the NP-complete problem BETWEENNESS[3] [10]:

INSTANCE: A finite set A, a collection T of ordered triples (a, b, c) of distinct elements from A.
QUESTION: Is there a total ordering $<$ on A such that for each $(a, b, c) \in T$, we have either $a < b < c$ or $c < b < a$?

Let us fix relations $R, R_1, R_2 \in \mathcal{A}$. Then we define $\Gamma(a, b, c, x, y)$ to be the following problem instance over the variables $\{a, b, c, x, y\}$:

$$\{xR_1a, xR_1b, xR_2c, yR_2a, yR_1b, yR_1c\}.$$

Let also $\Gamma_1 = \Gamma(a, b, c, x, y) \cup \{aRb, bRc\}$, $\Gamma_2 = \Gamma(a, b, c, x, y) \cup \{bRa, bRc, aR \cup R^{-1}c\}$, and $\Gamma_3 = \Gamma(a, b, c, x, y) \cup \{aRb, cRb, aR \cup R^{-1}c\}$.

Lemma 1 *Let $R \in \{(\mathsf{p}), (\mathsf{d}), (\mathsf{o}), (\mathsf{s})\}$, $R_1, R_2 \in \mathcal{A}$, and let Γ_1, Γ_2, Γ_3 be as above. If Γ_1 is satisfiable while Γ_2 and Γ_3 are not, then $\{R \cup R^{-1}, R_1, R_2\}$ is NP-complete.*

Proof. Polynomial-time reduction from BETWEENNESS. Let (A, T) be an arbitrary instance of the problem and construct an instance I of $\mathcal{A}$-SAT$(\{R \cup R^{-1}, R_1, R_2\})$ as follows:

(1) for each pair of distinct elements $a, b \in A$, add the constraint $aR \cup R^{-1}b$ to I; and

[3]This problem is also known as the TOTAL ORDERING PROBLEM [25].

(2) for each triple $(a, b, c) \in T$, introduce two fresh variables x, y and add $\Gamma(a, b, c, x, y)$ to I.

We will henceforth refer to the variables in I that corresponds to the set A as 'basic' variables and the other variables as 'auxiliary' variables.

Assume that I has a model M. First note that the constraints added in step (1) induce a total order (under the relation R) on the basic variables. Suppose now that there is a triple $(a, b, c) \in T$ such that the model M orders the variables $bRaRc$. This is impossible since $\Gamma(a, b, c, x, y) \subseteq \Theta$ for some auxiliary variables x and y which immediately implies that I is not satisfiable. We can analogously rule out all orders of a, b, c except $aRbRc$ and $cRbRa$ and (A, T) has a solution.

We assume now that there exists a total order $<$ on A that satisfies the triples in (A, T) and show how to construct a model of I. Assign intervals to the basic variables such that aRb iff $a < b$. Clearly, this satisfies all constraints added in step (1). The fact that $\Gamma(a, b, c, x, y) \cup \{aRb, bRc\}$ is satisfiable implies that $\Gamma(a, b, c, x, y) \cup \{cRb, bRa\}$ is satisfiable (by exchanging the value for x by the value for y and vice versa). Thus, for every triple $(a, b, c) \in T$, we can find values for the auxiliary variables that satisfies $\Gamma(a, b, c, x, y)$ and there exists a model of I. □

The restriction $R \in \{(\mathsf{p}), (\mathsf{d}), (\mathsf{o}), (\mathsf{s})\}$ is imposed to ensure that we get a natural total ordering in the reduction. Many other relations have this property but there are also relations without this property, e.g., (m) and (mm^{-1}).

Suppose $X \subseteq \mathcal{A}$ and I is an instance of $\mathcal{A}$-SAT(X). Let variables x, y be involved in I. Further, let $r \in \mathcal{A}$ be the relation defined as follows. A basic relation r' is included in r if and and only if the instance obtained from I by adding the constraint $xr'y$ is satisfiable. In this case, we say that r is *derived* from X. It can be easily checked that adding of a derived relation does not change the complexity of the problem. Looking now at the definition of the three operations of the interval algebra, it is easy to see that, generally, one can derive more relations from a given $X \subseteq \mathcal{A}$ than one can generate using these operations. On the other hand, derivation is essentially harder to manage while the operations of Allen's algebra give us the advantage of employing algebraic techniques. Therefore, in this paper, we use derivation only once in the proof of the following statement.

Proposition 2 *The following subsets of $\mathcal{A}$ are NP-complete:*
1) $\{(\mathsf{d}), (\mathsf{oo}^{-1})\}$, $\{(\mathsf{d}^{-1}), (\mathsf{pp}^{-1})\}$, *and* $\{(\mathsf{o}), (\mathsf{dd}^{-1})\}$.
2) $\{r\}$ *with* $(\mathsf{ods}^{-1}) \subseteq r \subseteq (\mathsf{pmods}^{-1}\mathsf{f}^{-1})$.

Proof. For the case (1), apply Lemma 1 with $R = (\mathsf{o})$, $R_1 = (\mathsf{d})$, $R_2 = (\mathsf{oo}^{-1})$, or with $R = (\mathsf{p})$, $R_1 = (\mathsf{d}^{-1})$, $R_2 = (\mathsf{pp}^{-1})$, or with $R = (\mathsf{d})$, $R_1 = (\mathsf{o})$, $R_2 = (\mathsf{dd}^{-1})$, respectively.

To prove part (2), let r_3 be the union of all basic relations except for $\equiv$ and s^{-1}, and consider the instance $\Gamma_4 = \{arb, brc, cra, drb, drc\}$ over the variables a, b, c, d. In the cases when $r = (\mathsf{ods}^{-1})$ or $r = (\mathsf{pmods}^{-1}\mathsf{f}^{-1})$, it can be shown that $\Gamma_4 \cup \{dr'a\}$ is satisfiable for every basic relation $r' \subseteq r_3$ but not satisfiable for any other choice of r'. Hence, for every r such that $\{r\}$ with $(\mathsf{ods}^{-1}) \subseteq r \subseteq (\mathsf{pmods}^{-1}\mathsf{f}^{-1})$, we can derive r_3 from r and we can also derive the relations $r_4 = r \cap r_3 = r - (\mathsf{s}^{-1})$ and $r_5 = r_4 \circ r_4$. It is easy to check that

$$(\mathsf{pmods}) \subseteq r_5 \subseteq (\mathsf{pmodsf}^{-1})$$

which implies that $r_5 \cap r^{-1} = (\mathsf{s})$. Furthermore, $(\mathsf{s}) \circ (\mathsf{s}^{-1}) = (\equiv \mathsf{ss}^{-1})$ and $r \circ r$ is the disequality relation, so the relation (ss^{-1}) can be obtained from the relation r. If $R = (\mathsf{s})$, $R_1 = r^{-1}$ and $R_2 = r$, then these relations satisfy the conditions of Lemma 1 so $\{(\mathsf{ss}^{-1}), r^{-1}, r\}$ is NP-complete. Since all of these relations can be derived from the single relation r, it follows that $\{r\}$ is NP-complete. □

4.2 The case $bas(\mathcal{S}) = \{\equiv, \mathsf{d}, \mathsf{d}^{-1}\}$

We will show that if $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ with $bas(\mathcal{S}) = \{\equiv, \mathsf{d}, \mathsf{d}^{-1}\}$, then $\mathcal{S} \subseteq \mathcal{S}_\mathsf{d}$, $\mathcal{S} \subseteq \mathcal{E}_\mathsf{d}$, or $\mathcal{S}$ is NP-complete.

Assume that $\mathcal{S}$ is not NP-complete. Then, as follows from Proposition 2(1), we have, in particular, that $(\mathsf{oo}^{-1}) \notin \mathcal{S}$ and $(\mathsf{pp}^{-1}) \notin \mathcal{S}$. We shall prove that then $\mathcal{S} \subseteq \mathcal{S}_\mathsf{d}$ or $\mathcal{S} \subseteq \mathcal{E}_\mathsf{d}$ holds. Before we give the proofs, let us note that $\nu = (\mathsf{d}^{-1}) \circ (\mathsf{d}) \in \mathcal{S}$.

Lemma 2 *The following inclusion holds:*

$$sym(\mathcal{S}) \subseteq \{r^* | (\mathsf{dd}^{-1}) \subseteq r\} \cup \{r^* | r \subseteq (\equiv \mathsf{ss}^{-1}) \cup \{r^* | r \subseteq (\equiv \mathsf{ff}^{-1})\}.$$

Proof. Suppose, for contradiction, that $sym(\mathcal{S})$ is not included in the above set. Suppose first that $(\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$ or $(\mathsf{ss}^{-1}\mathsf{ff}^{-1})$ belongs to $\mathcal{S}$. Then

$$r = (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1}) \circ (\mathsf{d}) = (\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \circ (\mathsf{d}) = (\mathsf{oo}^{-1}\mathsf{dsf}) \in \mathcal{S},$$

and we have $(\mathsf{oo}^{-1}) = r^* \in \mathcal{S}$, a contradiction.

Suppose now there is $r^* \in \mathcal{S}$ such that $r^* \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$ and $r^* \not\subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$. Consider $r_1 = (\mathsf{d}) \circ r^* \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dsf})$. It is easy to see that r_1^* is non-empty and $r_1^* \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1})$. Since $r_1^* \cap \nu$ cannot be equal to (oo^{-1}), we conclude that $r_1^* \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$. Then $(\mathsf{d}) \circ r_1^* = (\mathsf{pp}^{-1}) \in \mathcal{S}$, a contradiction. □

Lemma 3 *For every $r \in \mathcal{S}$ such that $r \not\subseteq (\equiv \mathsf{ss}^{-1})$ and $r \not\subseteq (\equiv \mathsf{ff}^{-1})$, the relation $r \cap (\mathsf{dd}^{-1})$ is non-empty.*

Proof. Suppose, for contradiction, there exists r satisfying the assumptions and with $r \cap (\mathsf{dd}^{-1})$ empty. Among such relations, choose r to be minimal with respect to inclusion. Then, since $\nu \in \mathcal{S}$, we have either $r \subseteq (\equiv \mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$, or $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ss}^{-1})$, or $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ff}^{-1})$.

Case 1. $r \subseteq (\equiv \mathsf{oo}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$

If $(\mathsf{o}) \subseteq r$ then $(\mathsf{o}^{-1}) \not\subseteq r$; otherwise r^* contradicts Lemma 2. Then $(\mathsf{d}^{-1}) \circ (\mathsf{o}) \subseteq (\mathsf{d}^{-1}) \circ r \subseteq (\mathsf{d}^{-1}) \circ (\equiv \mathsf{oss}^{-1}\mathsf{ff}^{-1})$, that is $(\mathsf{od}^{-1}\mathsf{f}^{-1}) \subseteq (\mathsf{d}^{-1}) \circ r \subseteq (\mathsf{oo}^{-1}\mathsf{d}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$. Since

r is minimal, we get $(\mathsf{o}) \subseteq r \subseteq (\mathsf{os}^{-1}\mathsf{f}^{-1})$. Further on, $(\mathsf{d}) \circ (\mathsf{o}) \subseteq (\mathsf{d}) \circ r \subseteq (\mathsf{d}) \circ (\mathsf{os}^{-1}\mathsf{f}^{-1})$, that is $(\mathsf{pmods}) \subseteq (\mathsf{d}) \circ r \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{oo}^{-1}\mathsf{dsf})$. Again, using minimality of r we conclude that $r = (\mathsf{o})$ which is impossible.

We may consider now that $r \subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$. If $(\mathsf{sf}) \subseteq r$ then $((\mathsf{d}^{-1}) \circ r)^* = (\mathsf{oo}^{-1}) \in \mathcal{S}$, a contradiction. If $r = (\equiv \mathsf{sf}^{-1})$ or $r = (\mathsf{sf}^{-1})$ then $r \cap ((\mathsf{d}) \circ r) = (\mathsf{s})$, a contradiction again. Thus, case 1 is impossible.

Case 2. $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ss}^{-1})$.
If $r \cap (\mathsf{pp}^{-1}) \neq \emptyset$ then $r_1 = r \cap ((\mathsf{d}) \circ r) \subseteq (\mathsf{pp}^{-1}\mathsf{m}^{-1})$. Further, $(\mathsf{d}) \circ r_1$ is a nonempty subrelation of (pp^{-1}), which contradicts our assumptions. If $(\mathsf{mm}^{-1}) \subseteq r$ then $((\mathsf{d}) \circ r)^* = (\mathsf{pp}^{-1})$, a contradiction again. It remains to consider the case $(\mathsf{m}) \subseteq r \subseteq (\equiv \mathsf{mss}^{-1})$. If $(\mathsf{s}^{-1}) \subseteq r$ then, again, $((\mathsf{d}) \circ r)^* = (\mathsf{pp}^{-1}) \in \mathcal{S}$. Otherwise $r_2 = r \circ r$ satisfies $(\mathsf{p}) \subseteq r_2 \subseteq (\equiv \mathsf{pms})$, and then $(\mathsf{p}) = r_2 \cap ((\mathsf{d}) \circ r_2) \in \mathcal{S}$, a contradiction.

Case 3. $r \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ff}^{-1})$.
Similar to Case 2. □

Lemma 4 *If* $(\mathsf{pmods}) \in \mathcal{S}$ *then* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$*; if* $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$.

Proof. We only prove the first statement, the second one is similar. First note that $r_{\mathsf{s}} \not\subseteq (\equiv \mathsf{ss}^{-1})$; in this case $r_{\mathsf{s}} = (\mathsf{s})$, since $(\mathsf{pmods}) \in \mathcal{S}$, which leads to a contradiction. Thus, the relations $r_{\mathsf{p}}, r_{\mathsf{m}}, r_{\mathsf{o}}, r_{\mathsf{s}}$ must be contained in (pmods) and, in view of Lemma 3, each of them contains (d). Hence, for every $r \in \mathcal{S}$, we have $(\mathsf{d}) \subseteq r$ whenever $r \cap (\mathsf{pmods})$ is non-empty. This condition precisely means that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$. □

Lemma 5 *If* $\mathcal{S}$ *contains a relation* $r \neq (\equiv)$ *such that* $r \subseteq (\equiv \mathsf{ss}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$*. If* $\mathcal{S}$ *contains a relation* $r \neq (\equiv)$ *such that* $r \subseteq (\equiv \mathsf{ff}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

Proof. The two cases are symmetric so we only consider the first one. Observe that $r \circ r^{-1} = (\equiv \mathsf{ss}^{-1})$ so we can assume that $r = (\equiv \mathsf{ss}^{-1})$. We have $(\mathsf{d}) \circ (\equiv \mathsf{ss}^{-1}) = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df})$ and the lemma follows from the preceding one. □

In view of Lemma 5, we may now assume that that, for any $r \in \mathcal{S}$ such that $r \neq (\equiv)$, $r \cap (\mathsf{dd}^{-1})$ is non-empty.

Lemma 6 *If* $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{o}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

Proof. Suppose first that $r_{\mathsf{o}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d})$. Then $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{o}}$ in order not to contradict Lemma 2. Since $r_{\mathsf{o}} \cap \nu = r_{\mathsf{o}}$, we have $r_1 = (\mathsf{od}) \subseteq r_{\mathsf{o}} \subseteq (\equiv \mathsf{odss}^{-1}\mathsf{ff}^{-1}) = r_2$. Consequently, $r_1 \circ (\mathsf{d}) \subseteq r_{\mathsf{o}} \circ (\mathsf{d}) \subseteq r_2 \circ (\mathsf{d})$, that is $(\mathsf{ods}) \subseteq r_{\mathsf{o}} \circ (\mathsf{d}) \subseteq (\mathsf{oo}^{-1}\mathsf{dsf})$. Then, by minimality of r_{o}, we have $(\mathsf{od}) \subseteq r_{\mathsf{o}} \subseteq (\mathsf{odsf})$. Finally,

$$(\mathsf{pmods}) = (\mathsf{d}) \circ (\mathsf{od}) \subseteq (\mathsf{d}) \circ r_{\mathsf{o}} \subseteq (\mathsf{d}) \circ (\mathsf{odsf}) = (\mathsf{pmods}),$$

that is $(\mathsf{pmods}) = (\mathsf{d}) \circ r_{\mathsf{o}} \in \mathcal{S}$, which implies, by Lemma 4, that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

Similar calculations show that if $r_{\mathsf{o}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$ then $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$, and hence $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$. □

Lemma 7 *If* $r_{\mathsf{s}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$ *or* $r_{\mathsf{f}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$, *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

Proof. The two cases are analogous so we only prove the first one. We have $(\mathsf{s}^{-1}) \not\subseteq r_{\mathsf{s}}$, otherwise r_{s}^* is non-empty but $r_{\mathsf{s}}^* \cap (\mathsf{dd}^{-1}) = \emptyset$ which contradicts the assumptions made. Thus, $(\mathsf{d}^{-1}\mathsf{s}) \subseteq r_{\mathsf{s}} \subseteq (\equiv \mathsf{oo}^{-1}\mathsf{d}^{-1}\mathsf{sff}^{-1})$.

Suppose $r_{\mathsf{s}} \cap (\mathsf{oo}^{-1})$ is non-empty. Then $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{o}}$ and $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$ by Lemma 6. Suppose to the contrary that $r_{\mathsf{s}} \cap (\mathsf{oo}^{-1}) = \emptyset$. Then, $(\mathsf{d}^{-1}\mathsf{s}) \subseteq r_{\mathsf{s}} \subseteq (\equiv \mathsf{d}^{-1}\mathsf{sff}^{-1})$ and

$$(\mathsf{od}^{-1}\mathsf{f}^{-1}) = (\mathsf{d}^{-1}) \circ (\mathsf{d}^{-1}\mathsf{s}) \subseteq (\mathsf{d}^{-1}) \circ r_{\mathsf{s}} \subseteq (\mathsf{d}^{-1}) \circ (\equiv \mathsf{d}^{-1}\mathsf{sff}^{-1}) = (\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}).$$

Therefore we have $(\mathsf{od}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$ or $(\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$, which implies $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{o}}$. By Lemma 6, $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$. □

Lemma 8 *If* $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{p}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

Proof. We consider only the case $r_{\mathsf{p}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d})$ since the case $r_{\mathsf{p}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$ is similar. As in the preceding lemmata, we have $(\mathsf{p}^{-1}) \not\subseteq r_{\mathsf{p}}$ because the opposite would contradict Lemma 2. So, let

$$(\mathsf{pd}) \subseteq r_{\mathsf{p}} \subseteq (\equiv \mathsf{pmm}^{-1}\mathsf{oo}^{-1}\mathsf{dss}^{-1}\mathsf{ff}^{-1}).$$

If $r_1 = r_{\mathsf{p}} \cap (\mathsf{oo}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$ is non-empty then using Lemmata 6 and 7 we easily get the required result; for example, $r_1 = (\mathsf{pod})$ implies $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{o}}$ and $r_1 = (\mathsf{pds}^{-1})$ implies $r_{\mathsf{s}^{-1}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d})$. Suppose now that $(\mathsf{pd}) \subseteq r_{\mathsf{p}} \subseteq (\equiv \mathsf{pmm}^{-1}\mathsf{dsf})$. Then

$$(\mathsf{pd}) = (\mathsf{d}) \circ (\mathsf{pd}) \subseteq (\mathsf{d}) \circ r_{\mathsf{p}} \subseteq (\mathsf{d}) \circ (\equiv \mathsf{pmm}^{-1}\mathsf{dsf}) = (\mathsf{pp}^{-1}\mathsf{d}).$$

Since $(\mathsf{pp}^{-1}\mathsf{d}) \notin \mathcal{S}$ (otherwise $(\mathsf{pp}^{-1}) \in \mathcal{S}$), we get $(\mathsf{pd}) = (\mathsf{d}) \circ r_{\mathsf{p}} \in \mathcal{S}$. Finally, $(\mathsf{pmods}) = (\mathsf{pd}) \circ (\mathsf{pd}) \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$ by Lemma 4. □

Lemma 9 *If* $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{m}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.

Proof. We consider only the case $r_{\mathsf{m}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d})$, the case $r_{\mathsf{m}} \cap (\mathsf{dd}^{-1}) = (\mathsf{d}^{-1})$ is similar.

As in the proof of the previous lemma, if $r_{\mathsf{m}} \cap (\mathsf{pp}^{-1}\mathsf{oo}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$ is non-empty then we get the required by Lemmata 6, 7 and 8. Now let $(\mathsf{md}) \subseteq r_{\mathsf{m}} \subseteq (\equiv \mathsf{mdsf})$. We have

$$(\mathsf{pd}) = (\mathsf{d}) \circ (\mathsf{md}) \subseteq (\mathsf{d}) \circ r_{\mathsf{m}} \subseteq (\mathsf{d}) \circ (\equiv \mathsf{mdsf}) = (\mathsf{pd}).$$

Thus, $(\mathsf{pd}) \in \mathcal{S}$, and, as in the previous lemma, $(\mathsf{pmods}) \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$. □

Lemma 10 *If a subalgebra $\mathcal{S}$ satisfies the assumptons of neither of Lemmata 5-9 then $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$.*

Proof. If a subalgebra $\mathcal{S}$ satisfies the assumptons of neither of Lemmata 5-9 then (d) is contained in all of the minimal relations r_{p}, r_{m}, r_{o}, and r_{s}. Therefore, every $r \in \mathcal{S}$ satisfies $r \cap (\mathsf{pmods}) \neq \emptyset \Rightarrow (\mathsf{d}) \subseteq r$, that is, $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{d}}$, and we get the required result. □

4.3 The case $bas(\mathcal{S}) = \{\equiv, \mathsf{o}, \mathsf{o}^{-1}\}$

We will now show that if $\mathcal{S}$ is a subalgebra of $\mathcal{A}$ with $bas(\mathcal{S}) = \{\equiv, \mathsf{o}, \mathsf{o}^{-1}\}$, then $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$, $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$ or $\mathcal{S}$ is NP-complete.

Assume that $\mathcal{S}$ is not NP-complete. Then, as follows from Proposition 2(1), we have, in particular, that $(\mathsf{dd}^{-1}) \notin \mathcal{S}$. We shall prove that then $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$ holds.

In the proofs, we will make frequent use of the fact that $\nu = (\mathsf{o}) \circ (\mathsf{o}^{-1}) \in \mathcal{S}$. Note also that $(\mathsf{pmo}) = (\mathsf{o}) \circ (\mathsf{o}) \in \mathcal{S}$ and the relation (pm) does not belong to $\mathcal{S}$ since, otherwise, $(\mathsf{p}) = (\mathsf{pm}) \circ (\mathsf{pm}) \in \mathcal{S}$. Therefore $r_{\mathsf{p}} = r_{\mathsf{m}} = (\mathsf{pmo})$ which implies that, for every $r \in \mathcal{S}$, if $r \cap (\mathsf{pmo}) \neq \emptyset$ then $(\mathsf{o}) \subseteq r$.

Lemma 11 *The following inclusion holds:*

$$sym(\mathcal{S}) \subseteq \{r^* | (\mathsf{oo}^{-1}) \subseteq r\} \cup \{r^* | r \subseteq (\equiv \mathsf{ss}^{-1})\} \cup \{r^* | r \subseteq (\equiv \mathsf{ff}^{-1})\}.$$

Proof. Suppose, for contradiction, that $sym(\mathcal{S})$ is not included in the above set. Suppose first that $(\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$ or $(\mathsf{ss}^{-1}\mathsf{ff}^{-1})$ belongs to $\mathcal{S}$. Then

$$r = (\mathsf{o}) \circ (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1}) = (\mathsf{o}) \circ (\mathsf{ss}^{-1}\mathsf{ff}^{-1}) = (\mathsf{pmodd}^{-1}\mathsf{sf}^{-1}) \in A,$$

and we have $(\mathsf{dd}^{-1}) = r^* \in \mathcal{S}$, a contradiction.

Suppose now that there exists $r^* \in \mathcal{S}$ such that $r^* \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$ and $r^* \not\subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$. If $r_1 = r^* \cap (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$ is non-empty then at least one of $r' = r_1 \cap (\mathsf{pmo}) \subseteq (\mathsf{pm})$ and $r'' = r_1 \cap (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}) \subseteq (\mathsf{p}^{-1}\mathsf{m}^{-1})$ is non-empty implying that $(\mathsf{p}) = r' \circ r'$ or $(\mathsf{p}^{-1}) = r'' \circ r''$ belongs to $\mathcal{S}$, a contradiction. Therefore $(\mathsf{dd}^{-1}) \subseteq r^*$ since $r^* \not\subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$. Now, it is easy to check that if $r_2 = ((\mathsf{o}) \circ r^*) \cap ((\mathsf{o}^{-1}) \circ r^*)$ then $r_2^* = (\mathsf{dd}^{-1}) \in \mathcal{S}$, a contradiction. □

Lemma 12 *For every $r \in \mathcal{S}$ such that $r \not\subseteq (\equiv \mathsf{ss}^{-1})$ and $r \not\subseteq (\equiv \mathsf{ff}^{-1})$, the relation $r \cap (\mathsf{oo}^{-1})$ is non-empty.*

Proof. Suppose, for contradiction, there exists r satisfying the assumptions and with $r \cap (\mathsf{oo}^{-1})$ empty. Among such relations, choose r to be minimal with respect to inclusion. Then, since, as noted above, $r \cap (\mathsf{pmo}) \neq \emptyset$ implies $(\mathsf{o}) \subseteq r$, we have $r \subseteq (\equiv \mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})$.

If $(\mathsf{d}) \subseteq r$ then $(\mathsf{d}^{-1}) \not\subseteq r$, since the opposite would contradict the preceding lemma. Then $(\mathsf{o}) \circ (\mathsf{d}) \subseteq (\mathsf{o}) \circ r \subseteq (\mathsf{o}) \circ (\equiv \mathsf{dss}^{-1}\mathsf{ff}^{-1})$, that is $(\mathsf{ods}) \subseteq (\mathsf{o}) \circ r \subseteq (\mathsf{pmodd}^{-1}\mathsf{sf}^{-1})$. Since r

is minimal, we get $(\mathsf{d}) \subseteq r \subseteq (\mathsf{dsf}^{-1})$. Furthermore, $(\mathsf{o}^{-1}) \circ (\mathsf{d}) \subseteq (\mathsf{o}^{-1}) \circ r \subseteq (\mathsf{o}^{-1}) \circ (\mathsf{dsf}^{-1})$, that is $(\mathsf{o}^{-1}\mathsf{df}) \subseteq (\mathsf{o}^{-1}) \circ r \subseteq (\mathsf{o}^{-1}\mathsf{dd}^{-1}\mathsf{s}^{-1}\mathsf{f})$. Again, using minimality of r we conclude that $r = (\mathsf{d})$ which is impossible.

We consider now the case $r \subseteq (\equiv \mathsf{ss}^{-1}\mathsf{ff}^{-1})$. If $(\mathsf{s}^{-1}\mathsf{f}) \subseteq r$ then $((\mathsf{o}) \circ r)^* = (\mathsf{dd}^{-1}) \in \mathcal{S}$, a contradiction. If $r = (\equiv \mathsf{sf})$ or $r = (\mathsf{sf})$ then $r \cap ((\mathsf{o}) \circ r) = (\mathsf{s})$, a contradiction again. □

The proofs of the following two lemmata are omitted since they are very similar to the proofs of Lemmata 4 and 5, respectively.

Lemma 13 *If* $(\mathsf{pmods}) \in \mathcal{S}$ *then* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$*; if* $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$.

Lemma 14 *If* $\mathcal{S}$ *contains a relation* $r \neq (\equiv)$ *such that* $r \subseteq (\equiv \mathsf{ss}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$*. If* $\mathcal{S}$ *contains a relation* $r \neq (\equiv)$ *such that* $r \subseteq (\equiv \mathsf{ff}^{-1})$*, then* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.

As a consequence of the preceding lemma, we can now assume that if $r \in \mathcal{S}$ and $r \neq (\equiv)$, then $r \cap (\mathsf{oo}^{-1})$ is non-empty.

Lemma 15 *If* $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{d}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ *or* $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.

Proof. Suppose first that $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}) = (\mathsf{o})$. Then $(\mathsf{dd}^{-1}) \not\subseteq r_{\mathsf{d}}$, since the opposite would contradict Lemma 11. It is the case that $r_{\mathsf{d}} \cap \nu = r_{\mathsf{d}}$ so $r_1 = (\mathsf{od}) \subseteq r_{\mathsf{d}} \subseteq (\equiv \mathsf{odss}^{-1}\mathsf{ff}^{-1}) = r_2$. Consequently, $(\mathsf{o}) \circ r_1 \subseteq (\mathsf{o}) \circ r_{\mathsf{d}} \subseteq (\mathsf{o}) \circ r_2$, that is $(\mathsf{pmods}) \subseteq (\mathsf{o}) \circ r_{\mathsf{d}} \subseteq (\mathsf{pmodd}^{-1}\mathsf{sf}^{-1})$. Then, by minimality of r_{d}, we have $(\mathsf{od}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{odsf}^{-1})$. Finally,

$$(\mathsf{pmods}) = (\mathsf{o}) \circ (\mathsf{od}) \subseteq (\mathsf{o}) \circ r_{\mathsf{d}} \subseteq (\mathsf{o}) \circ (\mathsf{odsf}^{-1}) = (\mathsf{pmods}),$$

that is $(\mathsf{pmods}) = (\mathsf{o}) \circ r_{\mathsf{d}} \in \mathcal{S}$, which implies, by Lemma 12, that $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.

Similar calculations show that if $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}) = (\mathsf{o}^{-1})$ then $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}$, and hence $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$. □

Lemma 16 *If* $r_{\mathsf{s}} \cap (\mathsf{oo}^{-1}) = (\mathsf{o}^{-1})$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$.

Proof. We have $(\mathsf{s}^{-1}) \not\subseteq r_{\mathsf{s}}$ since otherwise r_{s}^* is non-empty but $r_{\mathsf{s}}^* \cap (\mathsf{oo}^{-1}) = \emptyset$ which contradicts the assumptions made. Thus, $(\mathsf{o}^{-1}\mathsf{s}) \subseteq r_{\mathsf{s}} \subseteq (\equiv \mathsf{o}^{-1}\mathsf{dd}^{-1}\mathsf{sff}^{-1})$.

Suppose $r_{\mathsf{s}} \cap (\mathsf{dd}^{-1})$ is non-empty. Then, $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{d}}$ and $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$ or $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$ by Lemma 15. Note that $\mathcal{S} \not\subseteq \mathcal{E}_{\mathsf{o}}$ in view of $r_{\mathsf{s}} \cap (\mathsf{oo}^{-1}) = (\mathsf{o}^{-1})$. Therefore, we have $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$, as required.

Suppose now that $r_{\mathsf{s}} \cap (\mathsf{dd}^{-1}) = \emptyset$. Then $(\mathsf{o}^{-1}\mathsf{s}) \subseteq r_{\mathsf{s}} \subseteq (\equiv \mathsf{o}^{-1}\mathsf{sff}^{-1})$ and

$$(\mathsf{o}^{-1}) \circ (\mathsf{o}^{-1}\mathsf{s}) \subseteq (\mathsf{o}^{-1}) \circ r_{\mathsf{s}} \subseteq (\mathsf{o}^{-1}) \circ (\equiv \mathsf{o}^{-1}\mathsf{sff}^{-1}),$$

that is

$$(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df}) \subseteq (\mathsf{o}^{-1}) \circ r_{\mathsf{s}} \subseteq (\mathsf{o}^{-1}) \circ (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{dd}^{-1}\mathsf{s}^{-1}\mathsf{f}),$$

and by intersecting with ν we get

$$(\mathsf{o}^{-1}\mathsf{df}) \subseteq ((\mathsf{o}^{-1}) \circ r_{\mathsf{s}}) \cap \nu \subseteq (\mathsf{o}^{-1}\mathsf{dd}^{-1}\mathsf{sf})$$

Since $(\mathsf{dd}^{-1}) \notin \mathcal{S}$, we have $(\mathsf{o}^{-1}\mathsf{df}) \in \mathcal{S}$ or $(\mathsf{o}^{-1}\mathsf{dsf}) \in \mathcal{S}$. Therefore $(\mathsf{oo}^{-1}) \not\subseteq r_{\mathsf{d}}$ and, in view of $r_{\mathsf{s}} \cap (\mathsf{oo}^{-1}) = (\mathsf{o}^{-1})$, we get the result by the preceding lemma. □

Lemma 17 *If a subalgebra $\mathcal{S}$ satisfies the assumptons of neither of Lemmata 14-16 then $\mathcal{S} \subseteq \mathcal{E}_{\mathsf{o}}$.*

Proof. Suppose now that $\mathcal{S}$ satisfies neither of the conditions of Lemmata 14-16. Then (o) is contained in both r_{d} and r_{s}. Since, as we noted in the beginning of this Subsection, (o) is also contained in both r_{p} and r_{m}, we conclude that $\mathcal{S}$ is contained in $\mathcal{E}_{\mathsf{o}}$. □

4.4 The case $bas(\mathcal{S}) = \{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$ or $bas(\mathcal{S}) = \{\equiv, \mathsf{f}, \mathsf{f}^{-1}\}$

We will show that if $bas(\mathcal{S}) = \{\equiv, \mathsf{s}, \mathsf{s}^{-1}\}$, then either $\mathcal{S}$ is NP-complete or $\mathcal{S}$ is contained in one of the subalgebras $\mathcal{S}_{\mathsf{p}}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{E}^*$, $\mathcal{H}$, or in one of $\mathcal{A}_i$, $1 \leq i \leq 4$. By using the obvious symmetry between the relations (s) and (f), it immediately follows that if $bas(\mathcal{S}) = \{\equiv, \mathsf{f}, \mathsf{f}^{-1}\}$, then either $\mathcal{S}$ is NP-complete or contained in one of $\mathcal{E}_{\mathsf{p}}$, $\mathcal{E}_{\mathsf{d}}$, $\mathcal{E}_{\mathsf{o}}$, $\mathcal{S}^*$, $\mathcal{H}$, or $\mathcal{B}_i$, $1 \leq i \leq 4$.

Lemma 18 *If $\mathcal{S}$ contains the relation (od), then every $r \in \mathcal{S}$ satisfies condition 3) of $\mathcal{H}$.*

Proof. Arbitrarily choose $r \in \mathcal{S}$. Since $(\mathsf{od}) \in \mathcal{S}$, it follows that $r_{\mathsf{d}} = r_{\mathsf{o}} = (\mathsf{od})$ and $(\mathsf{o})^{\pm 1} \subseteq r \Leftrightarrow (\mathsf{d})^{\pm 1} \subseteq r$ $(*)$. Furthermore, $r_1 = (\mathsf{s}) \circ (\mathsf{od}) = (\mathsf{pmod}) \in \mathcal{S}$ so $r_{\mathsf{p}} \subseteq r_1$ and $r_{\mathsf{m}} \subseteq r_1$. Since $(\mathsf{pm}) \circ (\mathsf{pm}) = (\mathsf{p})$, we have $r_{\mathsf{p}} \not\subseteq (\mathsf{pm})$ and $r_{\mathsf{m}} \not\subseteq (\mathsf{pm})$, and therefore $r_{\mathsf{p}} \cap (\mathsf{od}) \neq \emptyset$ and $r_{\mathsf{m}} \cap (\mathsf{od}) \neq \emptyset$. By combining this with $(*)$, we have that $(\mathsf{od}) \subseteq r_{\mathsf{p}}$ and $(\mathsf{od}) \subseteq r_{\mathsf{m}}$. Thus, if $r \cap (\mathsf{pm}) \neq \emptyset$, then $(\mathsf{od}) \subseteq r$ and r satisfy condition 3) of $\mathcal{H}$. □

Lemma 19 *If $\mathcal{S}$ contains a non-trivial relation r' with $r' \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ff}^{-1})$ then $\mathcal{S}$ is included in one of $\mathcal{H}$, $\mathcal{E}^*$, or $\mathcal{S}_{\mathsf{p}}$, or else $\mathcal{S}$ is NP-complete.*

Proof. Case 1. $r' = (\mathsf{ff}^{-1})$
Since $(\mathsf{ff}^{-1}) \circ (\mathsf{s}) = (\mathsf{od}) \in \mathcal{S}$, it follows that any $r \in \mathcal{S}$ satisfies condition 3) of $\mathcal{H}$ by Lemma 18.

Suppose that $\mathcal{S} \not\subseteq \mathcal{H}$, i.e. some $r \in \mathcal{S}$ fails to satisfy condition 1) or condition 2) of $\mathcal{H}$. Then we may assume that

$$(\mathsf{s}^{-1}\mathsf{ff}^{-1}) \subseteq r \subseteq (\equiv \mathsf{pmodss}^{-1}\mathsf{ff}^{-1}) \text{ or } (\mathsf{ods}^{-1}) \subseteq r \subseteq (\equiv \mathsf{pmodss}^{-1}\mathsf{ff}^{-1}).$$

In both cases, multiplying the relations by (s) from the left we get

$$(\equiv \mathsf{pmodss}^{-1}) \subseteq (\mathsf{s}) \circ r \subseteq (\equiv \mathsf{pmodss}^{-1}).$$

Therefore $(\equiv \mathsf{pmodss}^{-1}) \in \mathcal{S}$, and

$$(\mathsf{pmods}^{-1}) = (\equiv \mathsf{pmodss}^{-1}) \cap ((\mathsf{ff}^{-1}) \circ (\equiv \mathsf{pmodss}^{-1})) \in \mathcal{S}.$$

Then $\mathcal{S}$ is NP-complete by Proposition 2(2).

Case 2. $r' \subseteq (\equiv \mathsf{ff}^{-1})$.
Multiplying r' and its inverse we get $(\equiv \mathsf{ff}^{-1})$, so we may assume that $r' = (\equiv \mathsf{ff}^{-1})$. If some relation $r_2 \in \mathcal{S}$ fails to satisfy condition 2) of $\mathcal{E}^*$ then $r_2 \cap r'$ is either $(\mathsf{f})^{\pm 1}$, which is impossible, or (ff^{-1}) going back to Case 1. Suppose now that each $r \in \mathcal{S}$ satisfies condition 2) of $\mathcal{E}^*$.

We have $r' \circ (\mathsf{s}) = (\mathsf{ods}) \in \mathcal{S}$. If $(\mathsf{od}) \in \mathcal{S}$ then $r' \cap ((\mathsf{s}^{-1}) \circ (\mathsf{od})) = (\mathsf{ff}^{-1}) \in \mathcal{S}$, which implies $\mathcal{S} \subseteq \mathcal{H}$. Suppose $(\mathsf{od}) \notin \mathcal{S}$. Then we have $r \cap (\mathsf{od})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r$ for every $r \in \mathcal{S}$. Assume a relation $r_3 \in \mathcal{S}$ does not satisfy condition 1) of $\mathcal{E}^*$, that is $r_3 \cap (\mathsf{pmod}) \neq \emptyset$ and $(\mathsf{s}) \not\subseteq r_3$. Since $(\mathsf{pmods}) = (\mathsf{s}) \circ r' \in \mathcal{S}$, we have $r_4 = r_3 \cap (\mathsf{pmods}) \in \mathcal{S}$ and $r_4 \subseteq (\mathsf{pm})$. This implies $r_4 \circ r_4 = (\mathsf{p})$, a contradiction. Therefore the relations in $\mathcal{S}$ must satisfy both conditions of $\mathcal{E}^*$, that is $\mathcal{S} \subseteq \mathcal{E}^*$.

Case 3. $r' \subseteq (\equiv \mathsf{mm}^{-1}\mathsf{ff}^{-1})$ and $r' \cap (\mathsf{mm}^{-1}) \neq \emptyset$.
Without loss of generality we may assume that $(\mathsf{m}) \subseteq r$. Then the relation $(r' \cap (r' \circ (\mathsf{s}^{-1}))$ is either (mm^{-1}) or (m). In the former case we get $(\mathsf{m}) = (\mathsf{mm}^{-1}) \cap ((\mathsf{mm}^{-1}) \circ (\mathsf{s}^{-1})) \in \mathcal{S}$, too. A contradiction.

Case 4. $r' \cap (\mathsf{pp}^{-1}) \neq \emptyset$.
Assume without loss of generality that $(\mathsf{p}) \subseteq r'$. Then $(\mathsf{p}) \subseteq (\mathsf{s}) \circ r' \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{ods})$. Further, $(\mathsf{p}) \subseteq r_5 = r' \cap ((\mathsf{s}) \circ r') \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1})$ and $(\mathsf{p}) \subseteq r_6 = r_5 \circ (\mathsf{s}^{-1}) \subseteq (\mathsf{pp}^{-1}\mathsf{m})$. If $(\mathsf{p}^{-1}) \not\subseteq r_6$ then $r_6 \circ r_6 = (\mathsf{p}) \in \mathcal{S}$, a contradiction. Otherwise $(\mathsf{pp}^{-1}) = r_6^* \in \mathcal{S}$. Then $(\mathsf{p}) \subseteq r \Leftrightarrow (\mathsf{p}^{-1}) \subseteq r$ holds for every $r \in \mathcal{S}$.

We have $(\mathsf{s}^{-1}) \circ (\mathsf{pp}^{-1}) = (\mathsf{pp}^{-1}\mathsf{modf}^{-1}) \in \mathcal{S}$. For every non-empty $r_7 \subseteq (\mathsf{modf}^{-1})$, except for $r_7 = (\mathsf{d})$, we have $((\mathsf{s}) \circ r_7) \cap (\mathsf{pp}^{-1}) = (\mathsf{p})$. Therefore no such r_7 belongs to $\mathcal{S}$. We conclude that, for any $r \in \mathcal{S}$, if $r \cap (\mathsf{modf}^{-1}) \neq \emptyset$ then $(\mathsf{p}) \subseteq r$, which means that $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{p}}$. □

We assume further that, for every non-trivial $r \in \mathcal{S}$, we have $r \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) \neq \emptyset$. In the next four lemmata we consider the case when $\mathcal{S}$ contains a non-trivial relation r' with $r' \cap (\mathsf{ss}^{-1}) = \emptyset$. In view of the assumption just made we may consider that r' is either r_{d} or r_{o}. It is not hard to verify that these four lemmata indeed cover all possible cases for r_{d} and r_{o}.

Lemma 20 *Suppose* $r_{\mathsf{d}} \cap (\mathsf{ss}^{-1}) = \emptyset$. *If* $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}) = (\mathsf{d})$ *or* $(\mathsf{dd}^{-1}) \subseteq r_{\mathsf{d}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$.

Proof. Case 1. $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}) = (\mathsf{d})$.
By assumption, the relation r_{d} satisfies condition $(\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\equiv \mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{dff}^{-1})$. Let r_1

be calculated as $r_{\mathsf{d}} \cap ((\mathsf{s}) \circ r_{\mathsf{d}})$. Then we have $(\mathsf{d}) \subseteq r_1 \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{d})$. By minimality of r_{d} we get $(\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{d})$. Calculating r_1 again, we get $(\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{pp}^{-1}\mathsf{m}^{-1}\mathsf{d})$. Since $(\mathsf{pp}^{-1}) \notin \mathcal{S}$, and since we have $(\mathsf{m}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}} \circ (\mathsf{s})$ only if $(\mathsf{p}^{-1}) \subseteq r_{\mathsf{d}}$, we may assume that r_{d} is either (pd), or $(\mathsf{p}^{-1}\mathsf{d})$, or $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{d})$. The first case is impossible because of $(\mathsf{pd}) \cap ((\mathsf{p}^{-1}\mathsf{d}^{-1}) \circ (\mathsf{s})) = (\mathsf{d})$. Let $(\mathsf{p}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{d})$. Then $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df}) = r_{\mathsf{d}} \circ (\mathsf{s}) \in \mathcal{S}$. Suppose $\mathcal{S}$ contains a non-empty subrelation r_2 of $(\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{f})$. Then $r_3 = r_{\mathsf{d}} \cap (r_2 \circ (\mathsf{s}^{-1}))$ is a non-empty subrelation of $(\mathsf{p}^{-1}\mathsf{m}^{-1})$ implying that $(\mathsf{p}^{-1}) = r_3 \circ r_3 \in \mathcal{S}$, a contradiction. Therefore , for every $r \in \mathcal{S}$, we have $r \cap (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{f}) \neq \emptyset \Rightarrow (\mathsf{d}) \subseteq r$, which means that $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$.

Case 2. $(\mathsf{dd}^{-1}) \subseteq r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1})$.
Suppose a relation $r_4 \in \mathcal{S}$ satisfies $(\mathsf{dd}^{-1}) \not\subseteq r_4$. Then it is not hard to verify that if $r_4 \not\subseteq (\equiv \mathsf{ss}^{-1})$ then either $((\mathsf{s}^{-1}) \circ r_4) \cap r_{\mathsf{d}}$ or $(r_4 \circ (\mathsf{s})) \cap r_{\mathsf{d}}$ contains exactly one of (d) and (d^{-1}) which contradicts the minimality of r_{d}. Thus, for every $r \in \mathcal{S}$ such that $r \not\subseteq (\equiv \mathsf{ss}^{-1})$, we have $(\mathsf{dd}^{-1}) \subseteq r$. This implies that $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$. □

Lemma 21 *Suppose* $r_{\mathsf{o}} \cap (\mathsf{ss}^{-1}) = \emptyset$. *If* $r_{\mathsf{o}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}) = (\mathsf{o})$ *or* $(\mathsf{oo}^{-1}) \subseteq r_{\mathsf{o}}$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{o}}$.

Proof. Similar to the previous lemma. □

Lemma 22 *If* $r_{\mathsf{d}} = r_{\mathsf{o}^{-1}}$ *and* $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) = (\mathsf{o}^{-1}\mathsf{d})$ *then* $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$.

Proof. As in the proof of Lemma 20, we can obtain $(\mathsf{o}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{pp}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df})$. If $(\mathsf{po}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}}$ then $(r_{\mathsf{d}} \circ (\mathsf{s}^{-1}))^* = (\mathsf{pp}^{-1}) \in \mathcal{S}$, a contradiction. Therefore we have $(\mathsf{o}^{-1}\mathsf{d}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df})$, and $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df}) = r_{\mathsf{d}} \circ (\mathsf{s}^{-1}) \in \mathcal{S}$. If some non-empty subrelation r_1 of $(\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{f})$ belongs to $\mathcal{S}$ then $r_2 = r_1 \cap (r_1 \circ (\mathsf{s}^{-1}))$ is a non-empty subrelation of $(\mathsf{p}^{-1}\mathsf{m}^{-1})$. Then $r_2 \circ r_2 = (\mathsf{p}^{-1})$ which contradicts our assumptions. Therefore, for every $r \in \mathcal{S}$, we have $r \cap (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{f}) \neq \emptyset \Rightarrow (\mathsf{d}) \subseteq r$, which means that $\mathcal{S} \subseteq \mathcal{S}_{\mathsf{d}}$. □

Lemma 23 *If* $r_{\mathsf{d}} = r_{\mathsf{o}}$ *and* $r_{\mathsf{d}} \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}) = (\mathsf{od})$ *then* $\mathcal{S} \subseteq \mathcal{H}$ *or* $\mathcal{S}$ *is NP-complete.*

Proof. As in the previous lemmata, it can be shown that $(\mathsf{od}) \subseteq r_{\mathsf{d}} \subseteq (\mathsf{pp}^{-1}\mathsf{mm}^{-1}\mathsf{od})$. Then $r_{\mathsf{d}}^* = \emptyset$ and $(\mathsf{pmod}) = (\mathsf{s}) \circ r_{\mathsf{d}} \in \mathcal{S}$. Further, $(\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1}) = ((\mathsf{s}^{-1}) \circ (\mathsf{pmod}))^* \in \mathcal{S}$ and $(\mathsf{od}) = (\mathsf{pmod}) \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1}) \in \mathcal{S}$. By Lemma 18, we know that every $r \in \mathcal{S}$ satisfies condition 3) of $\mathcal{H}$. Suppose some $r_1 \in \mathcal{S}$ does not satisfy condition 1) of $\mathcal{H}$. Then r_1 can be chosen so that

$$(\mathsf{s}^{-1}\mathsf{f}^{-1}) \subseteq r_1 \subseteq (\equiv \mathsf{pmodss}^{-1}\mathsf{ff}^{-1}) \text{ or } (\mathsf{os}^{-1}) \subseteq r_1 \subseteq (\equiv \mathsf{pmodss}^{-1}\mathsf{ff}^{-1}).$$

If $(\mathsf{f}) \subseteq r_1$ then $(\mathsf{ff}^{-1}) = (r_1 \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1}))^* \in \mathcal{S}$ which contradicts our assumptions. Further, if $(\mathsf{od}) \not\subseteq r_1$ then $r_1 \cap (\mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ff}^{-1}) = (\mathsf{f}^{-1}) \in \mathcal{S}$, a contradiction. Therefore

we may assume that $(\mathsf{ods}^{-1}\mathsf{f}^{-1}) \subseteq r_1 \subseteq (\equiv \mathsf{pmodss}^{-1}\mathsf{f}^{-1})$. Now it can checked that $r_2 = r_1 \cap (r_1^{-1} \circ (\mathsf{o}^{-1}\mathsf{d}^{-1}))$ satisfies $(\mathsf{ods}^{-1}\mathsf{f}^{-1}) \subseteq r_2 \subseteq (\mathsf{pmods}^{-1}\mathsf{f}^{-1})$. Then $\{r_2\}$ is NP-complete by Proposition 2(2). One can proceed similarly if condition 2) of $\mathcal{H}$ fails in $\mathcal{S}$. □

It remains to consider the case when each non-trivial $r \in \mathcal{S}$ satisfies $r \cap (\mathsf{ss}^{-1}) \neq \emptyset$.

Lemma 24 *If $r \cap (\mathsf{ss}^{-1}) \neq \emptyset$ for any non-trivial $r \in \mathcal{S}$ then $\mathcal{S} \subseteq \mathcal{A}_i$ for some $1 \leq i \leq 4$.*

Proof. Case 1. $r_{\mathsf{p}} \cap (\mathsf{ss}^{-1}) = (\mathsf{s}^{-1})$.
We have $(\mathsf{ps}^{-1}) \subseteq r_{\mathsf{p}}$ and $(\mathsf{s}) \not\subseteq r_{\mathsf{p}}$. Let $r_1 = (\mathsf{s}^{-1}) \circ r_{\mathsf{p}} \in \mathcal{S}$. Then it is easy to check that $(\mathsf{pmod}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}) \subseteq r_1$, and that $r_1 \cap (\equiv \mathsf{s}) = \emptyset$. It follows that $r_1 = (\mathsf{pmod}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$, since otherwise r_1^* is non-empty and $r_1^* \cap (\mathsf{ss}^{-1}) = \emptyset$. No non-empty subrelation of $(\mathsf{pmod}^{-1}\mathsf{f}^{-1})$ can belong to $\mathcal{S}$. Therefore, for any $r \in \mathcal{S}$, we have $r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \neq \emptyset \Rightarrow (\mathsf{s}^{-1}) \subseteq r$. Hence $\mathcal{S} \subseteq \mathcal{A}_1$.

Case 2. $r_{\mathsf{d}} \cap (\mathsf{ss}^{-1}) = (\mathsf{s}^{-1})$.
The proof is similar to Case 1; the only change is that $r_1 = r_{\mathsf{d}} \circ (\mathsf{s}^{-1})$, and we deduce that $r_1 = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{ds}^{-1}\mathsf{f})$, and, hence, $\mathcal{S} \subseteq \mathcal{A}_2$.

Case 3. $r_{\mathsf{o}} \cap (\mathsf{ss}^{-1}) = (\mathsf{s}^{-1})$.
In view of Cases 1 and 2 we may assume that $(\mathsf{os}^{-1}) \subseteq r_{\mathsf{o}} \subseteq (\equiv \mathsf{p}^{-1}\mathsf{mm}^{-1}\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{ff}^{-1})$. Let $r_1 = (\mathsf{s}^{-1}) \circ r_{\mathsf{o}} \in \mathcal{S}$. It is easy to check that r_1 satisfies $(\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}) \subseteq r_1 \subseteq (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{oo}^{-1}\mathsf{d}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$. Since $r_1^* \neq (\mathsf{oo}^{-1})$, we obtain $r_1 \subseteq (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$. It can straightforwardly be verified that if $r_1 \neq (\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$, then $(r_1 \circ r_1)^* \cap (\mathsf{pp}^{-1}\mathsf{ss}^{-1}) = (\mathsf{pp}^{-1}) \in \mathcal{S}$, which contradicts the assumptions made. Therefore $r_1 = (\mathsf{od}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1})$, and $(\mathsf{pmod}^{-1}\mathsf{s}^{-1}\mathsf{f}^{-1}) = r_1 \circ r_1 \in \mathcal{S}$. Therefore, for any $r \in \mathcal{S}$, $r \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \neq \emptyset$ implies $(\mathsf{s}^{-1}) \subseteq r$, that is $\mathcal{S} \subseteq \mathcal{A}_1$.

Case 4. $r_{\mathsf{m}} \cap (\mathsf{ss}^{-1}) = (\mathsf{s}^{-1})$.
Similarly to Case 3, we infer that $\mathcal{S} \subseteq \mathcal{A}_1$.

Case 5. (s) is contained in each of r_{p}, r_{d}, r_{o}, and r_{m}.
We have $r_{\mathsf{f}} \cap (\mathsf{ss}^{-1}) \neq \emptyset$. Then it follows that if $(\mathsf{s}) \subseteq r_{\mathsf{f}}$ then $\mathcal{S} \subseteq \mathcal{A}_3$. Otherwise, $(\mathsf{s}^{-1}) \subseteq r_{\mathsf{f}}$ and we have $\mathcal{S} \subseteq \mathcal{A}_4$. □

5 Discussion and Conclusions

We have proved that any so far unknown tractable fragment of Allen's algebra can contain no other basic relation than $(\equiv)$. We believe that, by extending the algebraic techniques used to establish this result, it will be possible to show that there are in fact no other forms of tractability in Allen's algebra. In other words, we conjecture that all maximal tractable subalgebras are already identified. However, the proof of this conjecture requires a more elaborate algebraic argument than the proofs in this paper, since in the general case we have no particular relation to start with.

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