Programming Research Group

SENSOR-BASED MOTION PLANNING IN SE(2) AND SE(3) VIA NONSMOOTH ANALYSIS

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Abstract

We study the sensor-based motion planning problem using the concepts of weak slope and generalised gradient from nonsmooth analysis. We give a nonsmooth analysis of the pointwise minima function $\phi_{min}(\mathbf{x}) = \min_i\{f_i(\mathbf{x})\}$ where each of the $f_i(\mathbf{x})$ are nonsmooth and non-convex, and in particular present a method to determine critical points of $\phi_{min}(\mathbf{x})$. We present a detailed nonsmooth analysis of the distance function of a free-floating body in $SE(2)$ and $SE(3)$ and use this to develop a novel algorithm for navigating in unknown environments for $SE(2)$.

1 Introduction

The "distance function" $D(\mathbf{q})$, usually expressed as the minimum of several "single object distance functions" $d_i(\mathbf{q})$, is commonly used in sensor-based motion planning as the basis for constructing a roadmap $\mathcal{R}$ of configuration space $\mathcal{C}$. However, all previous work that deals with realistic robot models has failed to appreciate that the distance function $d_i(\mathbf{q})$ between convex bodies is *not* everywhere differentiable. In fact, $d_i(\mathbf{q})$ is both nonsmooth and non-convex, so the usual tools of differential topology are not applicable. Nonetheless, a detailed understanding of the distance function is vital to the robotics community, and this paper presents the most realistic analysis to date. Our analysis is based on the nonsmooth notions of *weak slope* and *generalised gradient*, the former never having been used in robotics. These are powerful tools, especially when working in finite dimensional spaces, and we believe are of such general purpose that smoothness assumptions (as in [4]) may no longer be needed for a wide range of robotics problems.

This is not specifically a motion planning paper, but rather a detailed nonsmooth analysis of $d_i(\mathbf{q})$ and $D(\mathbf{q})$, with sensor-based planning used as an illustrative example. We present new results in nonsmooth analysis, the theory of distance functions, and sensor-based planning, so the paper should have wide appeal. The paper is rather heavy mathematically, so many of the less accessible proofs are postponed until the Appendix.

In sensor-based path planning, the robot $\mathcal{A}$ has no initial knowledge of it's environment (as in classical planning) and must use its sensors to navigate around obstacles $\mathcal{B}_i$. In Figure 1(a) $\mathcal{A}$ moves from configuration $\mathbf{q}_s$ to $\mathbf{q}_g$. This paper extends previous work in sensor-based planning to the case of free-floating, distance sensor equipped convex bodies in $SE(2)$ and $SE(3)$. Figure 1(b) shows our general approach. Scattered throughout a compact c-space, are a finite collection of compact *critical sets* K_c. Our algorithm constructs networks of segments consisting of critical configurations $\mathcal{R}_K$ and non-critical configurations $\mathcal{R}_{\neg K}$. The resulting roadmap $\mathcal{R} = \mathcal{R}_{\neg K} \cup \mathcal{R}_K$ is shown to be connected and is termed a *Nonsmooth Critical Set + Nonsmooth Generalised Voronoi Graph*, denoted NCrit+NGVG, since elements of $\mathcal{R}$ are comparable to GVG's [4].

In Section 2 we introduce tools from nonsmooth analysis [1, 5] and in Section 3 we apply these to $d_i(\mathbf{q})$ and $D(\mathbf{q})$. In Section 4 we present a technique to detect critical points of $D(\mathbf{q})$ and in Section 5 present a complete sensor-based planner for $SE(2)$. We introduce notation as required, bold denotes elements of a multidimensional space, and we use the general function symbol ϕ to present results not specific to $d_i(\mathbf{q})$ or $D(\mathbf{q})$.

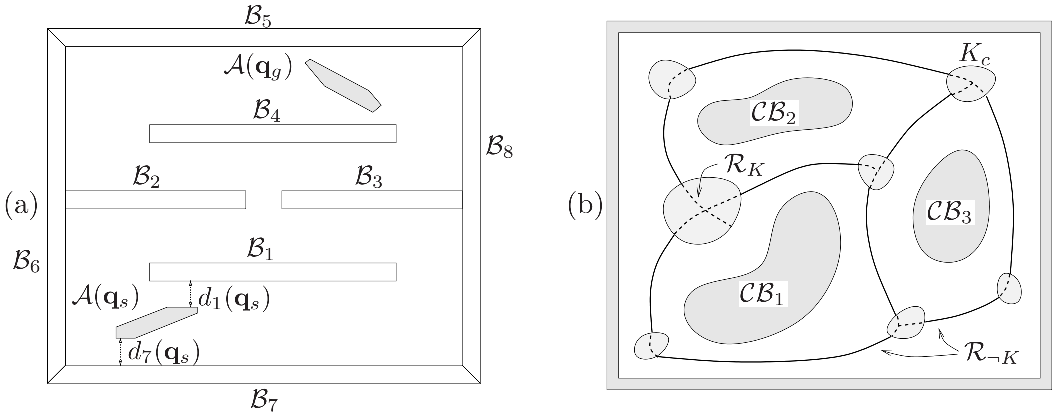


Figure 1: (a) Typical example in $SE(2)$. (b) The NCrit+NGVG in a 2D configuration space (doesn't match (a))

1.1 Previous Work in Motion Planning

Classical: The *visibility graph* [10] generates a roadmap for 2D (resp. 3D) c-space populated with polygonal (resp. polyhedral) obstacles $\mathcal{B}_i$. Techniques based on *Voronoi diagrams* [6] construct a roadmap which tends to maximise distance from $\mathcal{B}_i$'s. The *Silhouette* methods [2, 11] use the notion of *critical points* to construct a roadmap. In [2], roadmap segments are generated by recursively tracing critical points on the surface on c-space obstacles $\mathcal{CB}_i$ using a *sweep function*. In [11] the roadmap is generated by tracing the critical points of the *distance function* $D(\mathbf{q})$. The *Probabilistic Roadmap* [8] is constructed by placing a number of random points in c-space and joining then with a local planner that tests the feasibility of a path between adjacent points. Finally, the *EquiDistance Diagram* [9] is a network of curves equidistant to some number of $\mathcal{B}_i$'s, assuming $d_i(\mathbf{q})$ is smooth.

Sensor-Based: The first complete sensor based planner for arbitrary c-spaces was presented in [12]. This work was based on the incremental construction of one-dimensional *ridge curves* in freespace $\mathcal{C}_{free}$, based on local distance measurements. Although theoretically elegant, this work assumes $D(\mathbf{q})$ is a smooth function. Our work is most closely related to the notion of the *Generalised Voronoi Graph* [4] which assumes $d_i(\mathbf{q})$ is smooth, and [3] which was the first to address nonsmooth issues in sensor-based motion planning, albeit for a point robot.

2 Nonsmooth Analysis

In this Section we present definitions of the weak slope and generalised gradient, along with heuristic explanations of their meanings. In the following $B_\delta(\mathbf{x})$ denotes an open δ-ball. We start with the weak slope

Definition 0.1 ([1]) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be continuous. For every $\mathbf{x} \in \mathbb{R}^n$ we denote by $|d\phi|(\mathbf{x})$ the supremum of the σ's in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H}: B_\delta(\mathbf{x}) \times [0, \delta] \to \mathbb{R}^n$ such that (i) and (ii) are true

$$\begin{array}{ll} \text{(i)} & \forall \mathbf{v} \in B_\delta(\mathbf{x}), \forall t \in [0, \delta] : \operatorname{dist}(\mathcal{H}(\mathbf{v}, t), \mathbf{v}) \leq t \\ \text{(ii)} & \forall \mathbf{v} \in B_\delta(\mathbf{x}), \forall t \in [0, \delta] : \phi(\mathcal{H}(\mathbf{v}, t)) - \phi(\mathbf{v}) \leq -\sigma t \end{array} \tag{1}$$

$|d\phi|(\mathbf{x})$ is called the **weak slope** of ϕ at $\mathbf{x}$. ■

Corollary 0.1 $|d\phi|(\mathbf{x}) = \| \nabla\phi(\mathbf{x}) \|$ if $\nabla\phi(\mathbf{x})$ exists. ■

Heuristically, if $|d\phi|(\mathbf{x}) > 0$, we can map a nbhd($\mathbf{x}$) (nbhd() denotes neighbourhood) "downhill" continuously. Figure 2 shows "rubber sheets" centred at two points $\phi(\mathbf{x}_1)$ and $\phi(\mathbf{x}_2)$. We have $|d\phi|(\mathbf{x}_1) = 0$, since every point of the sheet cannot be stretched downhill without "poking a hole" in it (changing the topology). $|d\phi|(\mathbf{x}_2) > 0$ since we can stretch all points "downhill". Weak slope is used as the basis for determining critical points.

Definition 0.2 If $|d\phi|(\mathbf{x}) = 0$, then $\mathbf{x}$ is considered a **critical point** of ϕ. Other points are called **regular points**. A real number c is called a **critical value** of ϕ if $K_c \neq \varnothing$ (see below), and otherwise a **regular value** ■

We use these short-form subsets of $\mathbb{R}^n$

$$\begin{aligned} K_c &= \{\mathbf{x} \in \mathbb{R}^n \,;\, \phi(\mathbf{x}) = c, |d\phi|(\mathbf{x}) = 0\} && \text{(critical level set)} \\ L_c &= \{\mathbf{x} \in \mathbb{R}^n \,;\, \phi(\mathbf{x}) = c\} && \text{(level set)} \\ M_c &= \{\mathbf{x} \in \mathbb{R}^n \,;\, \phi(\mathbf{x}) \leq c\} && \text{(below level set)} \end{aligned}$$

and use K to denote the set of all critical points ($K = \cup K_c$). Sometimes $K_{\phi(\mathbf{x})}$ is subscripted to identify the function. The Palais-Smale condition (referred to as condition-C in some literature), is vital to establishing compactness of K_c, since it ensures no critical points at infinity and $\phi(K)$ is closed.

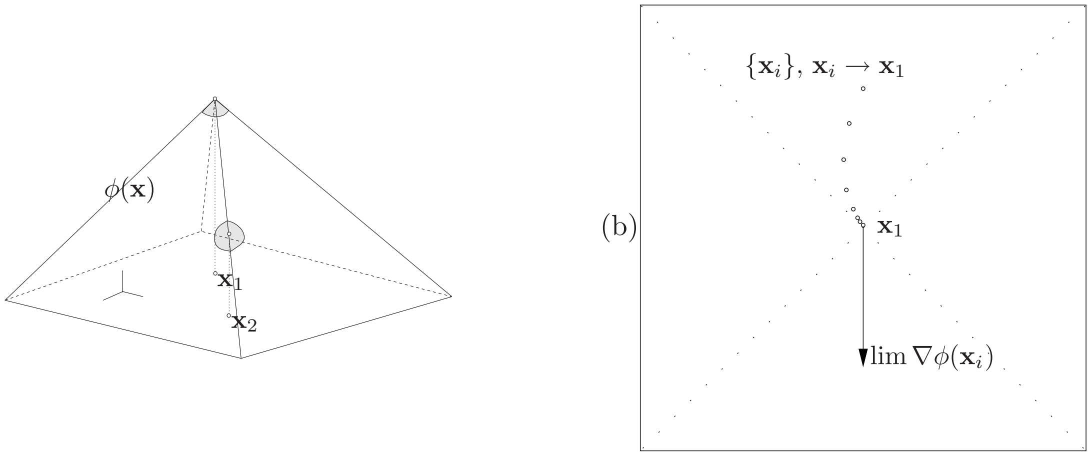
Definition 0.3 ϕ verifies the **Palais-Smale condition** (PS), if any sequence $\{\mathbf{x}_n\}$ satisfying $\lim_n \phi(\mathbf{x}_n) = c$ and $\lim_n |d\phi|(\mathbf{x}_n) = 0$ has a convergent subsequence. ■

We now present present the generalised gradient, which is constructed in terms of the generalised directional derivative.

Definition 0.4 Let $\phi: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function with $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$. The **generalised directional derivative**, denoted $\phi^\circ(\mathbf{x}; \mathbf{e})$, is defined as

$$\forall\, \mathbf{e} \in \mathbb{R}^n : \phi^\circ(\mathbf{x}; \mathbf{e}) = \limsup_{\mathbf{w} \to \mathbf{x}, t \to 0^+} \frac{\phi(\mathbf{w} + t\mathbf{e}) - \phi(\mathbf{w})}{t}$$ ■

$\phi^\circ(\mathbf{x}; \mathbf{e})$ is **upper semicontinuous** (u.s.c.) in $\mathbf{x}$,$\mathbf{e}$ [5]. So, when $\phi^\circ(\mathbf{x}; \mathbf{e}) < 0$, $\phi(\mathbf{x} + t\mathbf{e}) < \phi(\mathbf{x})$ for small enough t (ϕ tends "downhill" in direction $\mathbf{e}$). In fact ϕ tends "downhill" in direction $\mathbf{e}$ for all $\mathbf{y} \in$ nbhd($\mathbf{x}$). "uphill" implies $\phi^\circ(\mathbf{x}; \mathbf{e}) \geq 0$, but the converse is not true.

(a) 

(b)

Figure 2: (A) WEAK SLOPE IS VISUALISED USING "RUBBER SHEETS". (B) $\partial\phi(\mathbf{x}_1)$ IS CONVEX HULL OF GRADIENT VECTOR LIMITS. $\partial\phi(\mathbf{x}_1) = \{(0,-1),(1,0),(0,1),(-1,0)\}$

Definition 0.5 The **generalised gradient** of $\phi(\mathbf{x})$ in Definition 0.4, denoted $\partial\phi(\mathbf{x})$, is the set

$$\partial\phi(\mathbf{x}) = \{\boldsymbol{\alpha} \in \mathbb{R}^n : \phi^\circ(\mathbf{x};\mathbf{e}) \geq \langle \boldsymbol{\alpha}, \mathbf{e}\rangle \;\; \forall\, \mathbf{e} \in \mathbb{R}^n\} \qquad \blacksquare$$

Alternately, $\partial\phi(\mathbf{x})$ is given by

$$\partial\phi(\mathbf{x}) = \mathrm{co}\{\lim \nabla\phi(\mathbf{x}_i) : \mathbf{x}_i \to \mathbf{x}, \mathbf{x}_i \notin S, \mathbf{x}_i \notin \Omega_\phi\} \tag{2}$$

where $\Omega_\phi \subset \mathbb{R}^n$ is the set where $\nabla\phi(\mathbf{x})$ doesn't exist, S is a set of measure zero, and co$\{\}$ denotes convex hull. Figure 2(b) shows that $\partial\phi(\mathbf{x})$ is the convex hull of these gradient limits (as single limit $\lim \nabla\phi(\mathbf{x}_i)$ is shown for sequence $\{\mathbf{x}_i\}$). An important point is that $\partial\phi(\mathbf{x})$ is compact and convex [5].

The reader should sense from previous discussions that $|d\phi|(\mathbf{x})$ are $\partial\phi(\mathbf{x})$ are linked through the notion of "downhill". In $\mathbb{R}^n$ with ϕ Lipschitz we have[1] Theorem 0.1 which links the notions of weak slope and generalised gradient.

Theorem 0.1 $|d\phi|(\mathbf{x}) = 0$ iff $\mathbf{0} \in \partial\phi(\mathbf{x})$.
Proof See Appendix. $\blacksquare$

Note: All previous discussions have referred to $\mathbb{R}^n$, so we must assume $\mathcal{C}$ is charted to $\mathbb{R}^n$, in some manner.

2.1 A Nonsmooth Preimage Theorem

Preimage theorems for smooth functions are used to determine the dimension of roadmap segments in [4, 9]. To our knowledge, no nonsmooth preimage theorems appear in the

[1] $|d\phi|(\mathbf{x})$ and $\partial\phi(\mathbf{x})$ are actually defined in complete metric spaces and Banach spaces respectively, so Theorem 0.1 requires $\mathbb{R}^n$ as the domain. We assume $\mathcal{C}$ is charted to $\mathbb{R}^n$

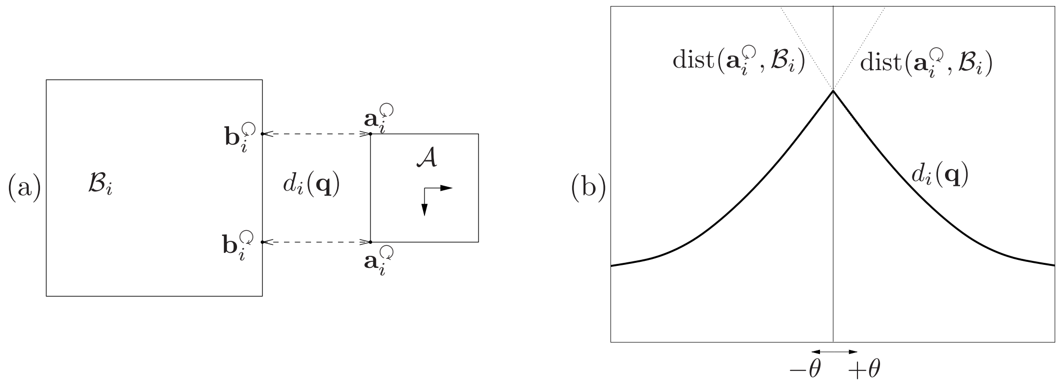


Figure 3: (A) A NONSMOOTH CONFIGURATION. (B)

literature, so we have developed one. Let $S \subset \mathbb{R}^m$ $(m \geq n)$ be locally homeomorphic to an open subset of $\mathbb{R}^n$.

Theorem 0.2 If $\mathbf{x} \in S$ is a regular point of $\phi\colon S \to \mathbb{R}$, with $\phi(\mathbf{x}) = y$, then the preimage $\phi^{-1}(y)$ has dimension $n-1$ in a nbhd of $\mathbf{x}$.
Proof See Appendix. ■

3 Nonsmooth Analysis of $\mathbf{d_i(q)}$ and $\mathbf{D(q)}$

We make an assumption that the obstacles $\mathcal{B}_i$ bound a compact subset of the workspace $\mathcal{W}$, and denote the resulting compact "reachable" c-space as $\mathcal{C}_r$. The "single object distance function" $d_i\colon\mathcal{C} \to \mathbb{R}$ between a (convex) robot $\mathcal{A}$ and $\mathcal{B}_i$ is

$$d_i(\mathbf{q}) = \min_{\mathbf{a}\in\mathcal{A}(\mathbf{q}),\mathbf{b}\in\mathcal{B}_i} \|\mathbf{a} - \mathbf{b}\|$$

and the "distance function" $D\colon\mathcal{C} \to \mathbb{R}$ is

$$D(\mathbf{q}) = \min_i \{d_i(\mathbf{q})\}$$

where i indexes a finite number of $\mathcal{B}_i$'s. We now show that a study of $d_i(\mathbf{q})$ requires the tools of nonsmooth analysis.

Proposition 0.1 $d_i(\mathbf{q})$ is nonsmooth and non-convex.
Proof Figure 3(a) shows $\mathcal{A}$ in a nonsmooth configuration (multiple closest points). $+\theta$ rotation results in *single* closest point $\mathbf{a}_i^{\circlearrowleft}$ and $-\theta$ gives $\mathbf{a}_i^{\circlearrowright}$, with point distance functions $\text{dist}(\mathbf{a}_i^{\circlearrowleft}(\mathbf{q}), \mathcal{B}_i)$ and $\text{dist}(\mathbf{a}_i^{\circlearrowright}(\mathbf{q}), \mathcal{B}_i)$ respectively. We can see $d_i(\mathbf{q})$ is nonsmooth and non-convex in Figure 3(b), which shows a "slice" of $d_i(\mathbf{q})$ through the θ coordinate. ■

To use the tools of nonsmooth analysis, we further require that no critical points occur at infinity. Theorem 0.3 establishes this.

Lemma 0.1 $D(\mathbf{q})$ is totally bounded (above and below).
Proof Trivial. ■

Lemma 0.2 $D(\mathbf{q})$ satisfies (PS).
Proof Take any sequence $\{\mathbf{q}_n\}$ in $\mathcal{C}_r$ with $\lim_n D(\mathbf{q}_n) = c$ and $\lim_n |dD|(\mathbf{q}_n) = 0$. By boundedness of $\mathcal{C}_r$, any subsequence $\{\mathbf{q}_n\}_k$ is bounded, so $\{\mathbf{q}_n\}$ admits a convergent subsequence, and $D(\mathbf{q})$ satisfies (PS). ■

Theorem 0.3 $K_{D(\mathbf{q})}$ is a compact set in $\mathcal{C}_r$.
Proof $D(\mathbf{q})$ is bounded (Lemma 0.1) and $D(K)$ is closed (Lemma 0.2), so $D(K)$ is compact. $\mathcal{C}_r$ is closed, so by the continuity of $D(\mathbf{q})$, $D^{-1}(K) = K$ is a closed set. K is compact since it is a closed subset of the compact metric space $\mathcal{C}_r$. ■

The compactness of K is required when showing K has a finite number of disjoint, connected components (see Section 4.1). In a nonsmooth setting, there is no guarantee of a finite number of critical *points*, which is usually facilitated by assuming ϕ is a smooth, Morse function.

3.1 Generalised Gradients $\partial \mathbf{d_i}(\mathbf{q})$ and $\partial \mathbf{D}(\mathbf{q})$

In this section we derive expressions for $\partial d_i(\mathbf{q})$ and $\partial D(\mathbf{q})$, and present a method to approximate $\partial d_i(\mathbf{q})$ from sensor measurements. To define $\partial d_i(\mathbf{q})$ and $\partial D(\mathbf{q})$ we must have $d_i(\mathbf{q})$ and $D(\mathbf{q})$ Lipschitz.

Lemma 0.3 $d_i(\mathbf{q})$ satisfies the Lipschitz condition $|d_i(\mathbf{q}_1) - d_i(\mathbf{q}_2)| \leq \| \mathbf{q}_1 - \mathbf{q}_2 \|$. $D(\mathbf{q})$ is also Lipschitz.
Proof See Appendix. ■

In Figure 4(a), let $\mathbf{a}_i$ be the closest point on $\mathcal{A}$ to $\mathcal{B}_i$ and let $\mathbf{b}_i$ be the closest point on $\mathcal{B}_i$. $\hat{\mathbf{a}}_i$ denotes the coordinates of $\mathbf{a}_i$ in $\mathcal{A}$'s coordinate frame $\mathcal{F}_{\mathcal{A}}$. We use the standard Euler angle parameterisation represented by R_ϕ, R_θ and R_ψ. $U = R_\phi R_\theta R_\psi \hat{\mathbf{a}}_i$ is vector $\hat{\mathbf{a}}_i$ in world frame $\mathcal{F}_{\mathcal{W}}$. x, y, z subscripts denote components of vectors in $\mathcal{W}$.

Theorem 0.4 $\nabla d_i(\mathbf{q})$ exists and is equal to the 6×1 vector

$$\nabla d_i(\mathbf{q}) = \frac{1}{d_i(\mathbf{q})} \begin{bmatrix} (\mathbf{a}_i - \mathbf{b}_i) \\ (\mathbf{U} \times (\mathbf{a}_i - \mathbf{b}_i))_z \\ (\mathbf{R}_\phi^{-1}(\mathbf{U} \times (\mathbf{a}_i - \mathbf{b}_i)))_y \\ (\mathbf{R}_\theta^{-1}\mathbf{R}_\phi^{-1}(\mathbf{U} \times (\mathbf{a}_i - \mathbf{b}_i)))_z \end{bmatrix} \tag{3}$$

iff there are single unique points $\mathbf{a}_i$ and $\mathbf{b}_i$ on $\mathcal{A}$ and $\mathcal{B}_i$ respectively, such that $\| \mathbf{a}_i - \mathbf{b}_i \| = d_i(\mathbf{q}) > 0$.
Proof Not included due to space restrictions. ■

In $SE(2)$, we simply have the 3×1 vector $\nabla d_i(\mathbf{q}) = [\mathbf{a}_i - \mathbf{b}_i, (\mathbf{U} \times (\mathbf{a}_i - \mathbf{b}_i))_z]^T$, where $U = R_\theta \hat{\mathbf{a}}_i$ and R_θ is the usual rotation matrix.

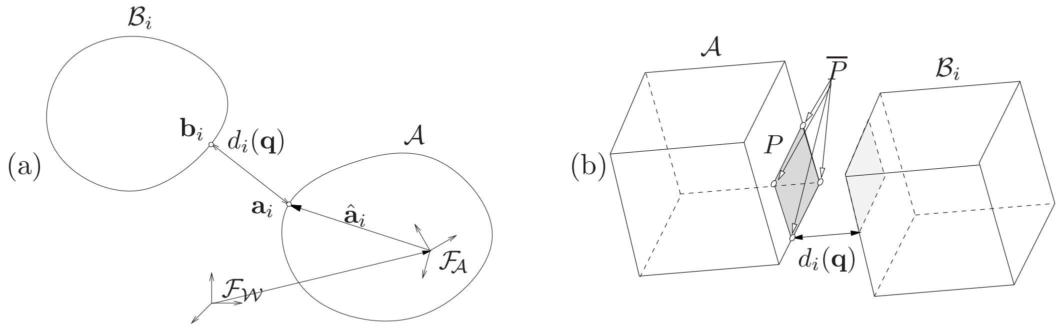


Figure 4: (A) NOTION USED TO DETERMINE $\nabla d_i(\mathbf{q})$. (B) P IS THE SET OF CLOSEST POINTS. $\overline{P}$ IS THE FINITE SUBSET OF SUPPORT POINTS.

By convexity of $\mathcal{A}$ and $\mathcal{B}_i$, when there are multiple closest points, there are an infinite number of closest points. Let these be denoted $\mathbf{a}_t(\mathbf{q}) \in \text{bndy}(\mathcal{A})$ (bndy() denotes boundary) with $t \in T$ (an abstract indexing space). $\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$ is the point distance function of a *single* point $\mathbf{a}_t(\mathbf{q})$. $\nabla\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$ is equivalent to$\nabla d_i(\mathbf{q})$ with $\mathbf{a}_t(\mathbf{q}) = \mathbf{a}_i$.

Lemma 0.4 The generalised gradient when there are multiple closest points is given by

$$\begin{aligned}\partial d_i(\mathbf{q}) = \text{co}\{\nabla\text{dist}(\mathbf{a}_t(\mathbf{q}),\mathcal{B}_i) \,:\, &\mathbf{a}_t(\mathbf{q}) \in \text{bndy}(\mathcal{A}),\\ &\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i) = d_i(\mathbf{q})\}\end{aligned} \tag{4}$$

Proof Note that $d_i(\mathbf{q}) = \min_t\{\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)\}$. Since $\mathcal{B}_i$ is convex, $\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$ is everywhere differentiable, so $\nabla\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$ exists. We also note $\min_i\{f_i\} = -\max_i\{-f_i\}$. By the smoothness of $\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$, $\partial d_i(\mathbf{q}) = \partial \max_t\{\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)\}$. Since $\dim(\mathcal{C})$ (dim() denotes dimension) is finite, the Lemma follows from Theorem 2.8.6 in [5]. ■

Let the set of closest points be denoted P (see Figure 4(b)). We have the following useful Lemma for approximating $\partial d_i(\mathbf{q})$ with a finite subset $\overline{P} \subset P$ of **support points**.

Lemma 0.5 $\partial d_i(\mathbf{q}) \subseteq \text{co}\{\nabla\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i); \mathbf{a}_t(\mathbf{q}) \in \overline{P}\}$
Proof See Appendix. ■

Finiteness is required for practical implementation by a computer, so Lemma 0.5 is of little use if P is not a convex polygon or line segment (a circle has an infinite number of support points), so we assume from now on that $\mathcal{A}$ and all $\mathcal{B}_i$'s are convex polytopes or polygons. Most real distance sensors (sonar, range sensor) are ideally able to supply $\overline{P}$.

4 Detection of Critical Points

Although $d_i(\mathbf{q})$ is in general nonsmooth, it does not have any nonsmooth (or smooth) critical points in $\mathcal{C}_{free}$. $\partial d_i(\mathbf{q})$ never contains the point $\mathbf{0}$ since $\mathbf{a} - \mathbf{b} \neq \mathbf{0}$ (see Equation 3). On the other hand, the distance function $D(\mathbf{q})$ is a pointwise minima function, which does have critical points.

Lemma 0.6 The generalised gradient $\partial\phi_{min}(\mathbf{x})$ of the pointwise minima function $\phi_{min}(\mathbf{x}) = \min_i\{f_i(\mathbf{x})\}$ is given by

$$\partial\phi_{min}(\mathbf{x}) \subset \text{co}\{\partial f_i(\mathbf{x}) : i \in I(\mathbf{x})\}$$

where $I(\mathbf{x})$ is the set of indices where $f_i(\mathbf{x}) = \phi_{min}(\mathbf{x})$.
Proof See Appendix. ■

The inclusion is a weak relationship, but we can still show the following for $\phi_{min}(\mathbf{x}) = \min_i\{f_i(\mathbf{x})\}$.

Theorem 0.5 Assuming $\mathbf{0} \notin \partial f_i(\mathbf{x})$ for all $i \in I(\mathbf{x})$, we have $\mathbf{0} \in \partial\phi_{min}(\mathbf{x})$ iff $\mathbf{0} \in \text{co}\{\partial f_i(\mathbf{x}) : i \in I(\mathbf{x})\}$ ■

Theorem 0.5 applies only if $\mathbf{0} \notin \partial f_i(\mathbf{x})$ as the functions $f_1(x) = x$ and $f_2(x) = |x|$ illustrate. We have $\partial f_1(0) = \{1\}$ and $\partial f_2(0) = [-1, 1]$, so $\text{co}\{\partial f_1(0), \partial f_2(0)\} = [-1, 1]$ which contains 0, but $\partial\phi_{min}(0) = \partial f_1(0) = \{1\}$ does not contain 0.

In [5], $\phi^\circ(\mathbf{x}; \mathbf{e})$ is shown to be the **support function** of $\partial\phi(\mathbf{x})$ meaning

$$\phi^\circ(\mathbf{x}; \mathbf{e}) = \max\{\langle\boldsymbol{\zeta}, \mathbf{e}\rangle \,:\, \boldsymbol{\zeta} \in \partial\phi(\mathbf{x})\} \tag{5}$$

So, to determine if $\mathbf{0} \in \partial\phi_{min}(\mathbf{x})$, we need only show that $\phi^\circ_{min}(\mathbf{x}; \mathbf{e}) \geq 0$ for all directions $\mathbf{e} \in \mathbb{R}^n$.

By definition of $\phi_{min}(\mathbf{x})$, if one $f_i(\mathbf{x})$ goes "downhill" in direction $\mathbf{e}$, so too does $\phi_{min}(\mathbf{x})$. $\partial f_i(\mathbf{x})$ provides a natural way to determine when $f_i^\circ(\mathbf{x}; \mathbf{e}) < 0$. For each $\partial f_i(\mathbf{x})$, we have an (open) cone of directions E_i where $f_i^\circ(\mathbf{x}; \mathbf{e}) < 0$ (see Figure 5(a)). Let $\neg E_i$ denote the set of directions not in E_i. Let $\{\mathbf{x}_k\}$ be an arbitrary sequence such that $\mathbf{x}_k \to \mathbf{x}$. By definition of u.s.c. and lim sup we have

$$f_i^\circ(\mathbf{x}; \mathbf{e}) \geq \limsup_{\mathbf{y}\to\mathbf{x}} f_i^\circ(\mathbf{y}; \mathbf{e}) \geq \limsup_{k\to\infty} f_i^\circ(\mathbf{x}_k; \mathbf{e}) \tag{6}$$

Lemma 0.7 For all $\mathbf{e} \in \neg E_i$, there exists a sequence $\{\mathbf{x}_k\}$ lying in $\mathbf{x} + E_i$ such that $\limsup_{k\to\infty} f_i^\circ(\mathbf{x}_k; \mathbf{e}) \geq 0$.
Proof See Appendix. ■

The preceding Lemma tells us we can construct sequences $\{\mathbf{x}_k\}$ lying on the "downhill" portions of $f_i(\mathbf{x})$ that go "uphill" for all directions $\mathbf{e} \in \neg E_i$. Since $\partial f_i(\mathbf{x})$ gives us the "downhill" directions, when $\text{co}\{\partial f_i(\mathbf{x}) \,:\, i \in I(\mathbf{x})\}$ contains the origin, there are "enough" sequences to show $f_i^\circ(\mathbf{x}; \mathbf{e}) \geq 0$ for all $\mathbf{e} \in \mathbb{R}^n$ (see Equation 6).

Proof of Theorem 0.5: If $\mathbf{0} \in \text{co}\{\partial f_i(\mathbf{x}) : i \in I(\mathbf{x})\}$, then $\cup_i \neg E_i = \mathbb{R}^n$. By Lemma 0.7 we can construct sequences $\{\mathbf{x}_k\}$ lying in each $\mathbf{x}+E_i$ ensuring $f_i^\circ(\mathbf{x}; \mathbf{e}) \geq 0$ for all $\mathbf{e} \in \neg E_i$. Since $\cup_i \neg E_i = \mathbb{R}^n$, we have $\phi^\circ_{min}(\mathbf{x}; \mathbf{e}) \geq 0$ for all $\mathbf{e} \in \mathbb{R}^n$. The other implication is immediate. ■

To detect if the current configuration $\mathbf{q}$ is critical, we simply measure the distance $d_i(\mathbf{q})$ to each of the visible $\mathcal{B}_i$'s, determine the n nearest $\mathcal{B}_i$'s, determine the individual generalised gradients $\partial d_i(\mathbf{q})$ and take the convex hull. If the convex hull contains $\mathbf{0}$, $\mathbf{q}$ is critical.

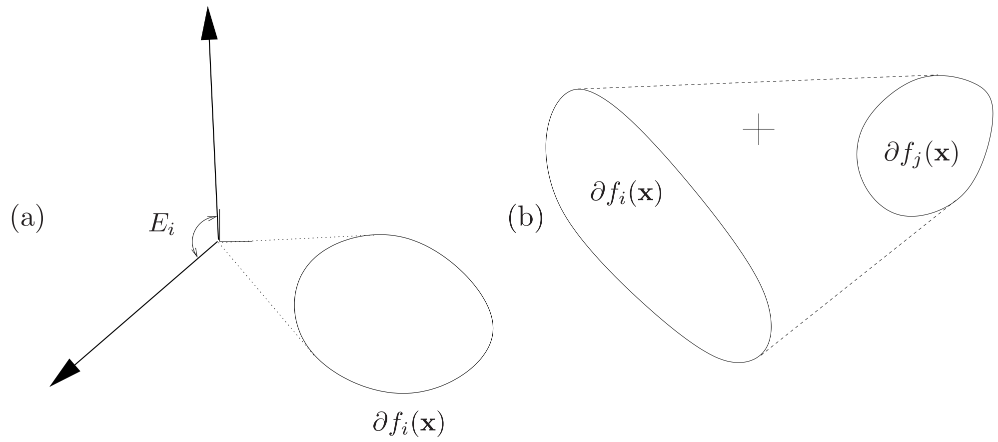


Figure 5: (a) Cone E_i of negative directions. (b) Convex hull contains the origin.

4.1 Finite Number of Components of $\mathbf{K}_{\mathbf{D(q)}}$

Let κ_d denote a disjoint, connected component of K. In order for a planner based on connecting critical sets κ_d to terminate, we must know *a priori* that K has a finite number of κ_d's. We show this by construction of an open nbhd of κ_d satisfying $\text{nbhd}(\kappa_d) \cap K = \kappa_d$ (κ_d is disconnected).

Definition 0.6 A set $A \subset \mathbb{R}^n$ is said to be **locally convex** at $\mathbf{a} \in A$ if there exists an ϵ-ball $B_\epsilon(\mathbf{a}) \subset \mathbb{R}^n$ such that $B_\epsilon(\mathbf{a}) \cap A$ is a convex set. ■

Lemma 0.8 There exists $\text{nbhd}(\kappa_d)$ such that $\text{nbhd}(\kappa_d) \cap K = \kappa_d$

Proof Take any point $\mathbf{q} \in \text{bndy}(\kappa_d)$. Our proof proceeds by constructing a normal vector $\mathbf{n}$ describing a hyperplane (after charting $\mathcal{C}$) through $\mathbf{q}$ such that $\langle \boldsymbol{\zeta}, \mathbf{n} \rangle \leq 0$ for all $\boldsymbol{\zeta} \in B_\epsilon(\mathbf{q}) \cap \kappa_d$ ($\mathbf{n}$ points away from convex set $B_\epsilon(\mathbf{q}) \cap \kappa_d$).

Set the translational terms $n_x = (\mathbf{b}_i - \mathbf{a}_i)_x$, $n_y = (\mathbf{b}_i - \mathbf{a}_i)_y$ and $n_z = (\mathbf{b}_i - \mathbf{a}_i)_z$. For any pure translational motion in a direction $\mathbf{e}$ satisfying $\langle \mathbf{e}, \mathbf{n} \rangle > 0$, we know $d_i^\circ(\mathbf{q}; \mathbf{e}) < 0$. By the u.s.c. of $d_i^\circ(\mathbf{q}; \mathbf{e})$, $d_i(\mathbf{q})$ (and $D(\mathbf{q})$) continue "downhill" for some distance $t\mathbf{e}$.

For angular coordinates, assume multiple closest points $\mathbf{a}_t(\mathbf{q})$. We have $d_i(\mathbf{q}) = \min_t \text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$ and $\partial d_i(\mathbf{q})$ given by Equation 4. As long as one $\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$ goes "downhill" in direction $\mathbf{e}$, so too do $d_i(\mathbf{q})$ and $D(\mathbf{q})$. If there is an element $\boldsymbol{\zeta} \in \partial d_i(\mathbf{q})$ with a negative θ coordinate, set $n_\theta = 1$ since all motions in direction $\mathbf{e}$ with positive θ coordinate will result in some $\text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$ going "downhill". If all $\boldsymbol{\zeta}$ have positive θ coordinates, set $n_\theta = -1$. If all $\boldsymbol{\zeta}$ have $\theta = 0$, set $n_\theta = \pm 1$.

By u.s.c of d_i°, for each $\mathbf{e}$, we have an open ended "segment" $l_\mathbf{e} = (\mathbf{q}, \mathbf{q} + t\mathbf{e})$, where $d_i(\mathbf{q_e}) < d_i(\mathbf{q})$ for all $\mathbf{q_e} \in l_\mathbf{e}$ for some small t. At point $\mathbf{q}$, select the minimum t over all directions in the halfspace and let $E_\mathbf{q} = \cup_\mathbf{e} l_\mathbf{e}$ be the open cone obtained as the union of all the "segments" $l_\mathbf{e}$ of length $t\mathbf{e}$. Set

$$\text{nbhd}(\kappa_d) = (\cup_{\mathbf{q} \in \text{bndy}(\kappa_d)} E_\mathbf{q}) \cup \kappa_d$$

(the union of all the cones $E_{\mathbf{q}}$ over the boundary of κ_d) which satisfies $\text{nbhd}(\kappa_d) \cap K = \kappa_d$ ■

The preceding Lemma implies that at any $\mathbf{q} \in \text{bndy}(\kappa_d)$, there is a halfspace of directions that decreases some $d_i(\mathbf{q})$.

Theorem 0.6 There are a finite number of κ_d's.
Proof Direct consequence of compactness of K (Theorem 0.3) and disconnectedness of κ_d's (Lemma 0.8). ■

Interestingly, we have the following (after charting a $\text{nbhd}(\mathbf{q})$) which proves useful in developing a planner.

Corollary 0.2 κ_d is locally convex at all points $\mathbf{q} \in \kappa_d$, and has Lebesgue measure zero in $\mathcal{C}$.
Proof Consequences of structure of $\text{nbhd}(\kappa_d)$ ■

5 Nonsmooth Planner in SE(2)

The NCrit+NGVG is defined in terms of $d_i(\mathbf{q})$. Like the GVG [4], subsets of configurations equidistant to two $\mathcal{B}_i$'s are intersected to obtain roadmap segments. However, with nonsmooth $d_i(\mathbf{q})$, we no longer deal with surfaces. We first describe the **Nonsmooth Generalised Voronoi Graph**, consisting of configurations equidistant to some number of $\mathcal{B}_i$'s, denoted $\mathcal{R}_{GVG} \subset \mathcal{R}$.

5.1 Nonsmooth Gen. Voronoi Graph $\mathcal{R}_{\mathbf{GVG}}$

Let the **two-equidistant set** be

$$\mathcal{S}_{ij} = \{\mathbf{q} \in SE(2) : d_i(\mathbf{q}) = d_j(\mathbf{q}) > 0\}$$

We will be using Theorem 0.2 to determine the dimension of $\mathcal{S}_{ij}$, but we must first determine when $(d_i - d_j)(\mathbf{q})$ is regular.

Lemma 0.9 $\mathbf{q}$ is a regular point of $(d_i - d_j)(\mathbf{q})$ if $\partial d_i(\mathbf{q}) \cap \partial d_j(\mathbf{q}) = \varnothing$.
Proof See Appendix. ■

We denote the subset of regular configurations the **invertible two-equidistant set**

$$\mathcal{IS}_{ij} = \{\mathbf{q} \in \mathcal{S}_{ij} : \partial d_i(\mathbf{q}) \cap \partial d_j(\mathbf{q}) = \varnothing\}$$

and by Theorem 0.2, $\dim(\mathcal{IS}_{ij}) = 2$.

To construct the NCrit+NGVG, we further define lower dimensional subsets of $\mathcal{S}_{ij}$. We start with the **three-equidistant set**

$$\mathcal{S}_{ijk} = \{\mathbf{q} \in \mathcal{S}_{ij} : d_i(\mathbf{q}) = d_j(\mathbf{q}) = d_k(\mathbf{q}) > 0\}$$

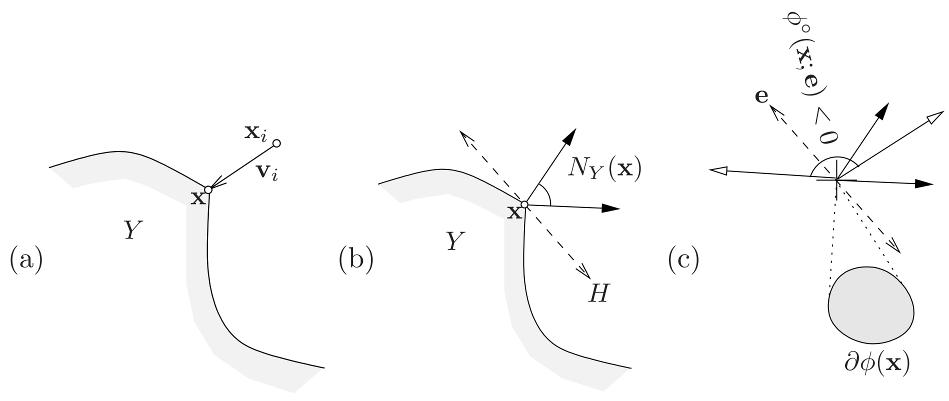


Figure 6: (A) CONSTRUCTION OF $N_Y(\mathbf{x})$. (B) SEGMENT OF bndy(Y) CONTAINING $\mathbf{x}$ AND $N_Y(\mathbf{x})$. (C) $\partial\phi(\mathbf{x})$ ALONG WITH $N_Y(\mathbf{x})$ AND DIRECTION $\mathbf{e} \in H$.

which is equivalent to the preimage $(d_i - d_k)^{-1}(\mathbf{0})$ of the function $(d_i - d_k) : \mathcal{S}_{ij} \to \mathbb{R}$. As noted in [5], the generalised gradient is defined in a *Banach space*, which $\mathcal{S}_{ij}$, $\mathcal{IS}_{ij}$ are not. We must determine regularity with the weak slope, which is valid for complete *metric spaces*. One more definition is required to facilitate to proof. The normal cone is the nonsmooth analogue of a normal vector. See [5] for more details.

Definition 0.7 The **normal cone** of a set $Y \subset \mathbb{R}^n$ at a point $\mathbf{x} \in Y$, denoted $N_Y(\mathbf{x})$ is given by (cl() denotes closure)

$$N_Y(\mathbf{x}) = \mathrm{cl}(\mathrm{co}\{\lambda \lim \tfrac{\mathbf{v}_i}{|\mathbf{v}_i|} : \lambda \geq 0, \mathbf{v}_i \perp Y \text{ at } \mathbf{x}_i, \\ \mathbf{x}_i \to \mathbf{x}, \mathbf{v}_i \to \mathbf{0}\})$$

where $\mathbf{v}_i = \mathbf{x}_i - \mathbf{x}$. See Figure 6(a), 6(b). ■

Denote the function ϕ *restricted to* Y with $\phi|_Y$.

Theorem 0.7 $\mathbf{x} \in \mathrm{bndy}(Y)$ is a regular point of $\phi|_{\mathrm{bndy}(Y)}$ if $\pm\boldsymbol{\zeta} + \boldsymbol{\nu} \neq \mathbf{0}$ for all $\boldsymbol{\zeta} \in \partial\phi(\mathbf{x})$ and $\boldsymbol{\nu} \in N_Y(\mathbf{x})$.
Proof Assume $\pm\boldsymbol{\zeta} + \boldsymbol{\nu} \neq \mathbf{0}$ for all $\boldsymbol{\zeta} \in \partial\phi(\mathbf{x})$ and $\boldsymbol{\nu} \in N_Y(\mathbf{x})$. We know then that $\mathbf{0} \notin \partial\phi(\mathbf{x})$ and so there must exist a direction $\mathbf{e} \in \mathbb{R}^n$ such that $\phi^\circ(\mathbf{x};\mathbf{e}) < 0$ and so ϕ tends "downhill" in direction $\mathbf{e}$. We need a continuous mapping $\mathcal{H}(\mathbf{v}, t)$ that maps points in $B_\delta(\mathbf{x}) \cap \mathrm{bndy}(Y)$ "downhill".

Define a hyperplane H such that $H \cap N_Y = H \cap -N_Y = \mathbf{0}$ which contains a direction $\mathbf{e}$ such that $\phi^\circ(\mathbf{x};\mathbf{e}) < 0$ (see Figure 6(b)). There exists a direction $\mathbf{e}$ for if there wasn't, we could find elements $\pm\boldsymbol{\zeta} + \boldsymbol{\nu} = \mathbf{0}$, a contradiction. Now by definition of the normal cone, the projection operator $\pi_H : \mathrm{bndy}(Y) \to H$ is a homeomorphism in a nbhd of $\mathbf{x}$. Define a continuous mapping in H (subscript $_H$), $\mathcal{H}_H : B_\delta(\mathbf{0}) \times [0, \delta] \to H$ with $\mathcal{H}_H(\mathbf{v}_H, t) = t\mathbf{e}$. In bndy($Y$), the mapping is $\mathcal{H}(\mathbf{v}, t) = \pi_H^{-1} \circ \mathcal{H}_H(\mathbf{v}_H, t)$ in a nbhd of $\mathbf{x}$. ■

There is no elegant "if and only if" proof regarding the criticality/regularity of points $\mathbf{q} \in \mathcal{C}$. The normal cone of a "below level set" M_y is determined from $\partial\phi(\mathbf{x})$

Lemma 0.10 If $\mathbf{x} \in \text{bndy}(M_y)$ is regular, $N_{M_y}(\mathbf{x}) \subseteq \{\lambda \partial \phi(\mathbf{x}) : \lambda \geq 0\}$
Proof Trivial. ■

We can now define the **invertible three-equidistant set**

$$\begin{aligned}\mathcal{IS}_{ijk} = \{\mathbf{q} \in \mathcal{IS}_{ij} : \pm\boldsymbol{\zeta} + \boldsymbol{\nu} \neq \mathbf{0}\ \forall\, \boldsymbol{\zeta} \in \partial(d_i - d_k)(\mathbf{q}),\\ \boldsymbol{\nu} \in \{\lambda \partial(d_i - d_j)(\mathbf{q})\ :\ \lambda \geq 0\}\}\end{aligned}$$

and the **invertible three-equidistant face**

$$\mathcal{IF}_{ijk} = \{\mathbf{q} \in \text{cl}(\mathcal{IS}_{ijk}) : d_i(\mathbf{q}) \leq d_l(\mathbf{q})\ \forall\, l \neq i, j, k\}$$

Which is the subset *closest* to $\mathcal{B}_i$, $\mathcal{B}_j$, $\mathcal{B}_k$. By Corollary 0.2, $\dim(\mathcal{IS}_{ijk}) = 1$, so $\mathcal{IF}_{ijk}$ are edges of NCRIT+NGVG.

In the smooth case [4], zero-dimensional subsets of $\mathcal{IS}_{ijk}$ equidistant to $\mathcal{B}_i$, $\mathcal{B}_j$, $\mathcal{B}_k$, $\mathcal{B}_l$ are defined when $\nabla(d_i - d_l)(\mathbf{q})$ is not a member of the subspace spanned by $\nabla(d_i - d_j)(\mathbf{q})$, $\nabla(d_i - d_k)(\mathbf{q})$. Analogues to transversality etc. exist in nonsmooth analysis, but to avoid choking the paper with definitions, we simply define the **invertible four-equidistant face**

$$\mathcal{IF}_{ijkl} = \mathcal{IF}_{ij} \cap \mathcal{IF}_{ik} \cap \mathcal{IF}_{kl}$$

and assume $\dim(\mathcal{IF}_{ijkl}) = 0$, so $\mathcal{IF}_{ijkl}$ are vertices of NCRIT+NGVG.

We use the notation $\mathcal{R}_{GVG} \subset \mathcal{R}$ to represent the set

$$\mathcal{R}_{GVG} = (\cup_{i,j,k} \mathcal{IF}_{ijk}) \cup (\cup_{i,j,k,l} \mathcal{IF}_{ijkl})$$

which is the set of 1D edges equidistant to three $\mathcal{B}_i$'s and the 0D vertices equidistant to four $\mathcal{B}_i$'s (see Figure 8(a)).

5.2 An Accessibility Retraction onto $\mathcal{R}_{\mathbf{GVG}}$

In this section we show it is possible to access $\mathcal{R}_{GVG}$ from any $\mathbf{q} \in \mathcal{C}_{free}$, using a retraction-like transformation. Retractions are useful in that they preserve the connectivity of their domain [10].

Lemma 0.11 For all $\mathbf{q} \in \mathcal{C}_{free}$, there exists a path $\tau(t)$ onto $\mathcal{R}_{GVG}$.
Proof Assume θ remains constant so all $d_i(\mathbf{q})$'s can be considered smooth with $\mathcal{C} = \mathbb{R}^2$. From start $\mathbf{q}_i$, find the closest $\mathcal{B}_i$ (assume w.l.o.g that $d_i(\mathbf{q}) < d_j(\mathbf{q})$ for all $j \neq i$) and move away in direction $\nabla d_i(\mathbf{q}) = \mathbf{a}_i - \mathbf{b}_i$, until equidistant to $\mathcal{B}_i$, $\mathcal{B}_j$ at $\mathbf{q}_{ij}$. We are assured that a $\mathcal{B}_j$ will be found since $\nabla d_i(\mathbf{q}) \neq \mathbf{0}$ for all $\mathbf{q} \in \mathcal{C}$ as $d_i(\mathbf{q})$ has no critical points. Find the next closest obstacle $\mathcal{B}_k$ and move towards it while maintaining equidistance with $\mathcal{B}_i$, $\mathcal{B}_j$, until equidistant to $\mathcal{B}_i$, $\mathcal{B}_j$, $\mathcal{B}_k$ at $\mathbf{q}_{ijk}$. This is done by projecting $\nabla d_i(\mathbf{q}) - \nabla d_k(\mathbf{q})$ onto the tangent space of $\mathcal{IS}_{ij}$ ($\pi_{T_{\mathbf{q}}\mathcal{IS}_{ij}}(\nabla d_i(\mathbf{q}) - \nabla d_k(\mathbf{q}))$).

If upon following the second step, $d_i(\mathbf{q}) = 0$ ("jammed"), $\pi_{T_{\mathbf{q}}\mathcal{IS}_{ij}}(\nabla d_i(\mathbf{q}) - \nabla d_k(\mathbf{q})) = \mathbf{0}$ or $\mathcal{B}_k$ disappears, then return to $\mathbf{q}_{ij}$ and find the next closest $\mathcal{B}_n \neq \mathcal{B}_k$ and repeat. We assume enough $\mathcal{B}_i$'s exist that this process terminates. ■

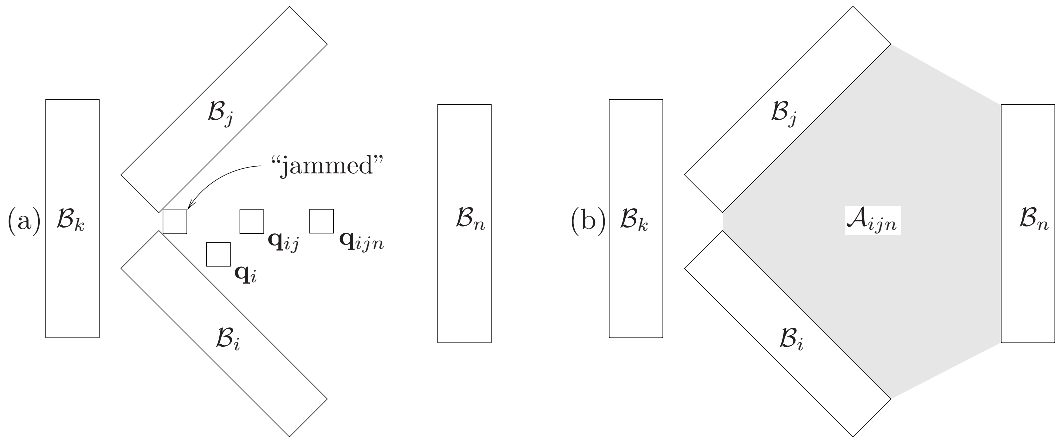


Figure 7: (A) ACCESSING $\mathcal{R}_{GVG}$. (B) ACCESSIBILITY REGION $\mathcal{A}_{ijn}$.

Note: this differs from the accessibility in [4] which ensures $\mathcal{A}$ can access a *specific* segment $\mathcal{IF}_{ijk}$ when $\mathcal{A}(\mathbf{q}_i)$ is closest to $\mathcal{B}_i$, $\mathcal{B}_j$ and $\mathcal{B}_k$. In Figure 7(a), $\mathcal{A}$ become "jammed" when moving towards $\mathcal{B}_k$.

Definition 0.8 An **accessibility region** $\mathcal{A}_{ijk}$ is the set of $\mathbf{q}_i$'s that result in $\mathbf{q}_{ijk} \in \mathcal{IF}_{ijk}$ when following path described in Lemma 0.11.

$$\mathcal{A}_{ijk} = \{\mathbf{q}_i \in \mathcal{C}_{free} : \mathbf{q}_{ijk} \in \mathcal{IF}_{ijk}\} \qquad \blacksquare$$

Figure 7(b) shows accessibility region $\mathcal{A}_{ijn}$. Trivially, $\cup_{i=1}^{i=n-2} \cup_{j=i+i}^{j=n-1} \cup_{k=j+1}^{k=n} \mathcal{A}_{ijk} = \mathcal{C}_{free}$. Here is our desired retraction.

Lemma 0.12 There is a continuous map $H : \mathcal{A}_{ijk} \times [0,1] \rightarrow \mathcal{IF}_{ijk}$ describing path from $\mathbf{q}_i \in \mathcal{A}_{ijk}$ to $\mathbf{q}_{ijk} \in \mathcal{IF}_{ijk}$.
Proof See [4] for similar proof. $\blacksquare$

It is important to note that H is not a global retraction, but still satisfies the "retraction-like" properties in [10].

Lemma 0.13 $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{A}_{ijk}$ are path connected in $\mathcal{A}_{ijk}$ iff $H(\mathbf{q}_1, 1)$, $H(\mathbf{q}_2, 1)$ are path connected in $\mathcal{IF}_{ijk}$.
Proof Basic property of retractions. $\blacksquare$

Lemma 0.13 tells us we can plan within $\mathcal{A}_{ijk}$ using only $\mathcal{IF}_{ijk}$, but we still require proof that paths exists between adjacent accessibility regions. The **critical set graph** $\mathcal{R}_K$ connects some cases.

5.3 Critical Set Graph $\mathcal{R}_\mathbf{K}$

We now define the graph $\mathcal{R}_K$ consisting of critical configurations. We know $K \subset \mathcal{IS}_{ij}$ since $\mathcal{A}$ must be equidistant to at least two $\mathcal{B}_i$'s to be critical. As well, in addition to

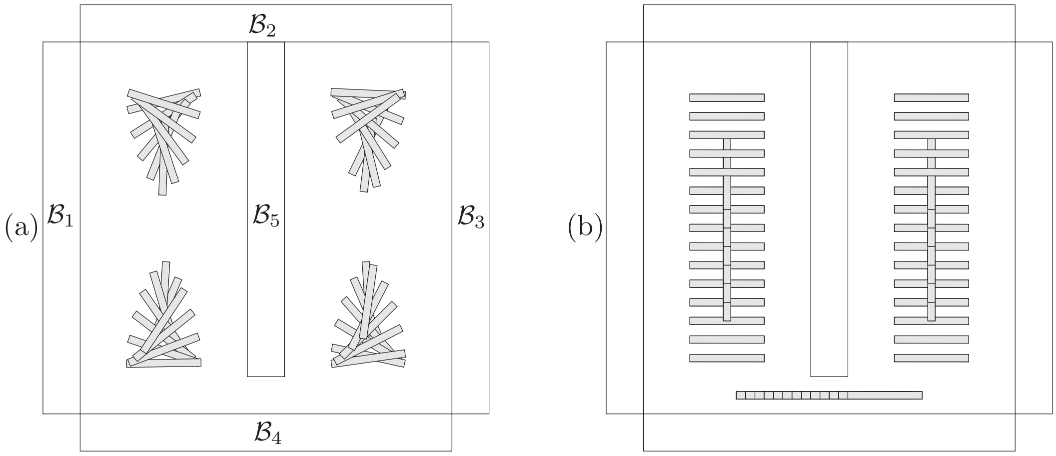


Figure 8: (a) $\mathcal{R}_{GVG}$ isn't connected to portions of (b) $\mathcal{R}_K$.

$d_i(\mathbf{q}) - d_j(\mathbf{q}) = 0$ for all $\mathbf{q} \in \mathcal{IS}_{ij}$, we have the necessary condition that $d_j(\mathbf{q}) = c$ for all $\mathbf{q} \in K_c$. Define the function $f_j : \mathcal{IS}_{ij} \to \mathbb{R}$ as

$$f_j(\mathbf{q}) = d_j(\mathbf{q}) - c$$

Level sets of f_j represent motion that doesn't change $d_j(\mathbf{q})$, and since $d_j(\mathbf{q}) = c$ is only a necessary condition for $\mathbf{q} \in K$, we have $K_c \subset f_j^{-1}(\mathbf{0})$.

Lemma 0.14 $\dim(K) \leq 1$
Proof Recall $\dim(\mathcal{IS}_{ij}) = 2$. We have $\partial f_j(\mathbf{q}) = \partial d_j(\mathbf{q})$, so trivially $\pm\partial f_j(\mathbf{q}) \cap N_{\mathcal{IS}_{ij}}(\mathbf{q}) = \varnothing$. Applying Theorem 0.7, we see that $\mathbf{q}$ is regular and so Theorem 0.2, $\dim(f_j^{-1}(\mathbf{0})) = 1$. Since $d_j(\mathbf{q}) = c$ is only a necessary condition for $\mathbf{q} \in K$, we have $\dim(K) = \dim(\kappa_d) \leq 1$. ■

In $SE(2)$ we have the unique result that $\mathcal{R}_K = K$. Figure 8(b) shows configurations from five κ_d's. By Corollary 0.2 κ_d is locally convex, and by Lemma 0.14 $\dim(\kappa_d) \leq 1$, so the κ_d is either a line segment or a point.

5.4 Connecting it all Together

We can see in Figure 8 that $\mathcal{R}_K \cup \mathcal{R}_{GVG}$ does not define a fully connected roadmap $\mathcal{R}$. However, we do know $\mathcal{R}_{GVG}$ is sufficient to connect critical sets K_{ijk} equidistant to $\mathcal{B}_i$, $\mathcal{B}_j$, $\mathcal{B}_k$.

Lemma 0.15 $K_{ijk} \subset (\mathcal{R}_{GVG} \cap K)\ \forall\ i, j, k \in \{1, .., n\}$
Proof Trivial. ■

More importantly is the existence of a path from critical sets K_{ij} equidistant to two obstacles $\mathcal{B}_i$, $\mathcal{B}_j$ onto $\mathcal{R}_{GVG}$. This path is based on $\mathcal{A}$'s **diameter**

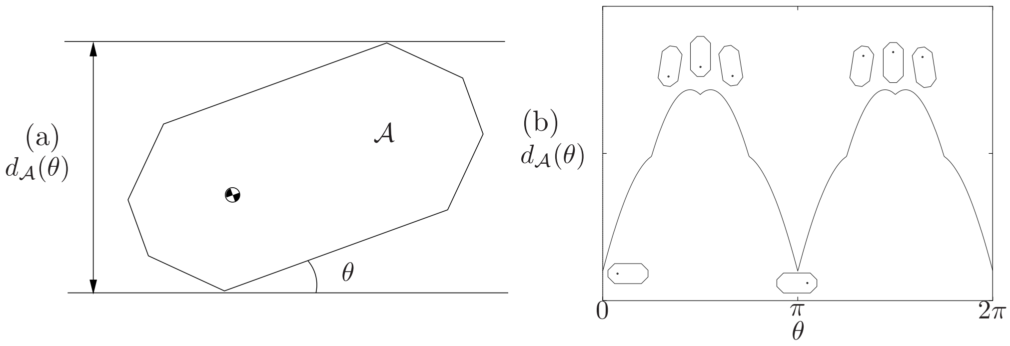


Figure 9: (A) CONSTRUCTION OF $d_{\mathcal{A}}(\theta)$. (B) EXTREMA OF $d_{\mathcal{A}}(\theta)$.

Definition 0.9 The **diameter function** of $\mathcal{A}$, $d_{\mathcal{A}} : S^1 \to \mathbb{R}$, maps the orientation θ to distance $d_{\mathcal{A}}(\theta)$ between two parallel supporting lines. (See Figure 9(a)) ■

The structure connecting K_{ij} to $\mathcal{R}_{GVG}$ is termed a **diameter function extrema edge**, denoted $\mathcal{D}_{ij}^{\vartheta}$, where ϑ is a local minima/maxima of $d_{\mathcal{A}}(\theta)$ (see Figure 9(b)). For each $\mathbf{q} \in \mathcal{D}_{ij}^{\vartheta}$, $d_i(\mathbf{q}) = d_j(\mathbf{q})$ and $\mathcal{A}$ is oriented so the support lines are perpendicular to $\mathbf{ab} = (\mathbf{a}_i - \mathbf{b}_i) - (\mathbf{a}_j - \mathbf{b}_j)$, or $(\mathbf{a}_i - \mathbf{b}_i)$ if $\mathbf{ab} = \mathbf{0}$. In the first case, this is represented by the preimage $f_{ij}^{-1}(\mathbf{0})$ of $f_{ij} : \mathcal{IS}_{ij} \to \mathbb{R}$ given by

$$f_{ij} = \tan^{-1}(\mathbf{ab}_y/\mathbf{ab}_x) - \theta - \vartheta + \pi/2$$

Figure 10(a) shows two $\mathcal{D}_{ij}^{\vartheta}$'s. We now show $\mathcal{D}_{ij}^{\vartheta}$ is an edge of NCRIT+NGVG.

Lemma 0.16 $\mathcal{D}_{ij}^{\vartheta}$ is one dimensional.
Proof We must show when $f_{ij}(\mathbf{q})$ is regular. Using the Chain Rule in [5], we have

$$\partial f_{ij}(\mathbf{q}) = (1 + (\mathbf{ab}_y/\mathbf{ab}_x)^2)^{-1} \partial(\mathbf{ab}_y/\mathbf{ab}_x) - [\mathbf{0}, 1]^T$$

By the Quotient Rule in [5], $\partial(\mathbf{ab}_y/\mathbf{ab}_x)$ has x, y elements (don't need θ coordinates to show regularity)

$$[-\mathbf{ab}_y, \mathbf{ab}_x, \times]^T$$

so $\partial f_{ij}(\mathbf{q})$ is "perpendicular" to $N_{\mathcal{IS}_{ij}}$ in the x, y coordinates, and $f_{ij}(\mathbf{q})$ is regular on $\mathcal{IS}_{ij}$. Since $\dim(\mathcal{IS}_{ij}) = 2$, by Corollary 0.2, $\dim(\mathcal{D}_{ij}^{\vartheta}) = 1$. A similar proof exists for $\mathbf{ab} = \mathbf{0}$ case. ■

Since our goal is to connect all κ_d's with $\mathcal{R}$, we need to show $\mathcal{D}_{ij}^{\vartheta}$ are sufficient to connect K_{ij} to $\mathcal{R}_{GVG}$. Lemma 0.17 establishes this.

Lemma 0.17 $K_{ij} \subset (\cup_{\vartheta} \mathcal{D}_{ij}^{\vartheta} \cap K)$
Proof Not included due to space restrictions. ■

And finally we show that some $\mathcal{D}_{ij}^{\vartheta}$ is sufficient to connect two adjacent accessibility regions $\mathcal{A}_{ijk}$, $\mathcal{A}_{ijl}$.

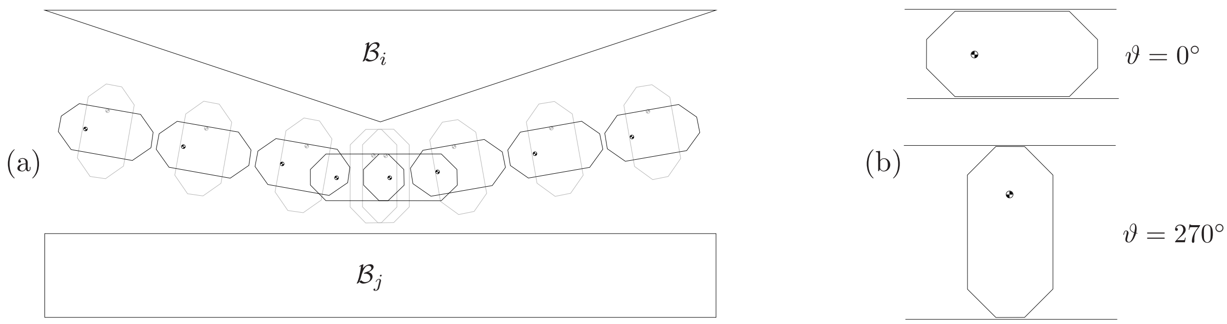


Figure 10: (A) TWO DIAMETER EXTREMA EDGES $\mathcal{D}_{ij}^{\vartheta_1}$ AND $\mathcal{D}_{ij}^{\vartheta_2}$. (B) VALUES OF ϑ_1, ϑ_2.

Lemma 0.18 If a path $\tau(t)$ exists between $\mathcal{IF}_{ijk}$, $\mathcal{IF}_{ijl}$, then $\mathcal{D}_{ij}^{\vartheta}$ connects $\mathcal{IF}_{ijk}$, $\mathcal{IF}_{ijl}$ for some ϑ.
Proof This is entirely self evident. If a path $\tau(t)$ exists, then the distance between $\mathcal{B}_i$, $\mathcal{B}_j$ is greater than $\min_\theta d_{\mathcal{A}}(\theta)$, and certainly one of the $\mathcal{D}_{ij}^{\vartheta}$'s passes from $\mathcal{IF}_{ijk}$ to $\mathcal{IF}_{ijl}$. ∎

We denote the the set of all diameter extrema edges $\mathcal{R}_D$

$$\mathcal{R}_D = \cup_{i,j,\vartheta} \mathcal{D}_{ij}^{\vartheta}$$

and let NCRIT+NGVG = $\mathcal{R}_{GVG} \cup \mathcal{R}_K \cup \mathcal{R}_D$. By Lemmas 0.15 and 0.17, NCRIT+NGVG connects and contains all κ_d's. By Lemmas 0.13 and 0.18, NCRIT+NGVG represents the connectivity of $\mathcal{C}_{free}$ and can thus be used for complete sensor-based planning.

6 Conclusions and Future Work

This paper has a presented a study of distance functions using the weak slope and generalised gradient from nonsmooth analysis. The notions of criticality and regularity were presented in a nonsmooth context, and an approach to detect critical configurations from sensor data, despite the nonsmooth/non-convex nature of the distance function, was presented. The concept of the critical point pervades the motion planning literature, so our nonsmooth characterisation is of great utility to the motion planning community.

Our sensor-based algorithm demonstrates the utility of the approach and shows that smoothness assumptions are not necessary. Future work includes developing techniques to follow $\mathcal{R}_{GVG}$, $\mathcal{R}_K$ and $\mathcal{R}_D$ based on nonsmooth control theory, clearly defining the advantages/disadvantages of nonsmooth analysis over simply assuming smoothness, and extending to $SE(3)$. Our characterisation of critical points of the distance function is highly geometric in nature, so we believe any algorithms in $SE(3)$ (or other c-space) will be specific to the c-space considered, rather than an application of a general algorithm for arbitrary c-spaces.

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Appendix

Proof of Theorem 0.1 This is not true in general (i.e. infinite dimensional domains), since by Theorem 2.17 [7] we have

$$|d\phi|(\mathbf{x}) \geq \min\{\|\boldsymbol{\alpha}\|: \boldsymbol{\alpha} \in \partial\phi(\mathbf{x})\}$$

which only gives one implication direction. Recall Eq. 2

$$\partial\phi(\mathbf{x}) = \mathrm{co}\{\lim \nabla\phi(\mathbf{x}_i) : \mathbf{x}_i \to \mathbf{x}, \mathbf{x}_i \notin S, \mathbf{x}_i \notin \Omega_\phi\}$$

Let the minimum norm element of $\partial\phi(\mathbf{x})$ be $\boldsymbol{\zeta}$. Now, there must exist a sequence of points $\{\mathbf{x}_k\}$ with $\mathbf{x}_k \to \mathbf{x}$ such that $\lim\limits_{\mathbf{x}_k \to \mathbf{x}} \nabla\phi(\mathbf{x}_k) = \boldsymbol{\zeta}$. By the continuity of the norm operator $\lim\limits_{\mathbf{x}_k \to \mathbf{x}} \| \nabla\phi(\mathbf{x}_k) \| = \| \boldsymbol{\zeta} \|$. Within any delta neighbourhood $B_\delta(\mathbf{x})$, by Corollary 0.1, there

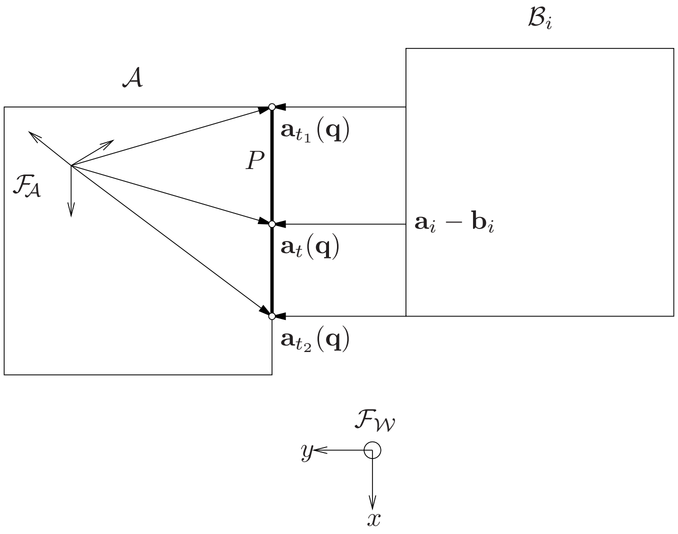

Figure 11: FIGURE 4(B) FROM DIFFERENT ANGLE.

exist points $\mathbf{x}_k$ where $|d\phi|(\mathbf{x}_k) = \| \nabla\phi(\mathbf{x}_k) \|$ so we must have $|d\phi|(\mathbf{x}) \leq \| \nabla\phi(\mathbf{x}_k) \|$ for all $\mathbf{x}_k \in B_\delta(\mathbf{x})$. Since $\lim\limits_{\mathbf{x}_k \to \mathbf{x}} \| \nabla\phi(\mathbf{x}_k) \| = \| \boldsymbol{\zeta} \|$ we have

$$|d\phi|(\mathbf{x}) \leq \min\{\| \boldsymbol{\alpha} \| : \boldsymbol{\alpha} \in \partial\phi(\mathbf{x})\} = \| \boldsymbol{\zeta} \|$$

so the theorem follows. ■

In order to prove Theorem 0.2, we need some preliminaries. We use Theorem 0.8 to develop a nonsmooth preimage Theorem 0.9.

Theorem 0.8 (Local Inversion Theorem [5]) If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ (ϕ is vector valued) is locally Lipschitz and each element of the generalised Jacobian[2] $\partial\phi(\mathbf{x})$ is invertible, then ϕ maps a nbhd of $\mathbf{x} \in \mathbb{R}^n$ homeomorphically onto a nbhd of $y = \phi(\mathbf{x})$. ■

Definition 0.10 The **canonical submersion** $\pi : \mathbb{R}^k \to \mathbb{R}^l$ is the standard projection of $\mathbb{R}^k$ onto $\mathbb{R}^l$ for $k \geq l$, in which $(x_1, \cdots, x_k) \to (x_1, \cdots, x_l)$. ■

Theorem 0.9 If $y \in \mathbb{R}$ is a regular value of $\phi : \mathbb{R}^n \to \mathbb{R}$, then the preimage $\phi^{-1}(y)$ is a set with dimension $n - 1$.
Proof If y is a regular value, then $\mathbf{0} \notin \partial\phi(\mathbf{x})$ for all $x \in L_y$. Consider local parameterisation diagram (l.p.d.) (a)

[2] The generalised Jacobian is an analogue of the generalised derivative for vector valued functions. See [5] for details.

$$\text{(a)}\quad \begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R} \\ I\uparrow & & \uparrow I \\ \mathbb{R}^n & \xrightarrow{g} & \mathbb{R} \end{array} \qquad\qquad \text{(b)}\quad \begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R} \\ G^{-1}\uparrow & & \uparrow I \\ \mathbb{R}^n & \xrightarrow{\pi} & \mathbb{R} \end{array}$$

We seek to modify g and apply the Local Inversion Theorem. Define $G:\mathbb{R}^n \to \mathbb{R}^n$ by $G(\mathbf{x}) = (g(\mathbf{x}), x_2, \cdots, x_n)$.

We know from [5] that $\partial G(\mathbf{x}) \subset \partial g(\mathbf{x}) \times \cdots \times \partial x_n(\mathbf{x})$, where the latter denotes the set of all matrices whose i^{th} row belongs to generalised gradient of the coordinate function. Since $\partial g(\mathbf{x})$ is invertible, we can have a change of coordinates so that $\partial g(\mathbf{x})$ always has a nonzero x_1 coordinate. The elements of $\partial G(\mathbf{x})$ have the full rank form (G^{-1} exists)

$$\begin{bmatrix} \partial g(\mathbf{x})_1 & \partial g(\mathbf{x})_2 & \cdots & \partial g(\mathbf{x})_n \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

By construction, $g = \pi \circ G$, so $g \circ G^{-1}$ is the canonical submersion π, so l.p.d.(b) commutes. Locally, $\phi(\mathbf{x})$ is equivalent to π, so select local coordinates around $\mathbf{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$ so $\phi(x_1, \cdots, x_n) = x_1$ and y corresponds to 0. Let V denote the nbhd of $\mathbf{x}$ in coordinate system $(x_1, \cdots, x_n)$. Then $\phi^{-1}(y) \cap V$ is the set of points where $x_1 = 0$, a relatively open subset of $\phi^{-1}(y)$ having dimension $n-1$. By the arbitrariness of $\mathbf{x}$, $\dim(\phi^{-1}(y)) = n-1$. ■

Proof of Corollary 0.2 If $\mathbf{x}$ is a regular point of S, then there exists a continuous transformation $\mathcal{H}(\mathbf{v}, t)$ satisfying Equations 2 in a neighbourhood $B_\delta(\mathbf{x})$ ($X = S$ in the notation of Equation 2). Let $G : S \to \mathbb{R}^n$ be a local homeomorphism between S and $\mathbb{R}^n$. Since $\mathcal{H}(\mathbf{v}, t)$ is continuous, so is $G \circ \mathcal{H}$ and thus $G(\mathbf{x}) \in \mathbb{R}^n$ is a regular point of $\phi_G = G^{-1} \circ \phi$ where $\phi_G(G(\mathbf{x})) = y$. By Theorem 0.2 $\phi_G^{-1}(y)$ has dimension $n-1$ in a neighbourhood of $G(\mathbf{x})$, and since homeomorphism preserves dimension, $G^{-1}(\phi_G^{-1}(y)) \cap B_\delta(\mathbf{x}) = \phi^{-1}(y) \cap B_\delta(\mathbf{x})$ has dimension $n-1$. ■

Proof of Lemma 0.3 Assume that the metric $d(\mathbf{q}_1, \mathbf{q}_2)$ on $\mathcal{C}$ is the so called "maximum distance metric"

$$d(\mathbf{q}_1, \mathbf{q}_2) = \max_{\mathbf{a} \in \mathcal{A}} \| \mathbf{a}(\mathbf{q}_1) - \mathbf{a}(\mathbf{q}_2) \|$$

which corresponds to the maximum distance moved by any of the points in the robot.

$\mathbf{a}_1(\mathbf{q}_1)$ is the closest point on $\mathcal{A}$ in $\mathbf{q}_1$, and $\mathbf{a}_1(\mathbf{q}_2)$ is the location of that same point in $\mathbf{q}_2$. Let ϵ be given, so $d_i(\mathbf{q}_1) \geq \| \mathbf{a}_1(\mathbf{q}_1) - \mathbf{b_1} \| - \epsilon$, and

$$\begin{aligned} d_i(\mathbf{q}_2) &\leq \| \mathbf{a}_1(\mathbf{q}_2) - \mathbf{b}_1 \| \\ &\leq \| \mathbf{a}_1(\mathbf{q}_2) - \mathbf{a}_1(\mathbf{q}_1) \| + \| \mathbf{a}_1(\mathbf{q}_1) - \mathbf{b}_1 \| \\ &\leq \| \mathbf{q}_1 - \mathbf{q}_2 \| + d_i(\mathbf{q}_1) + \epsilon \end{aligned} \tag{7}$$

By arbitrariness of ϵ, and since we can switch roles of $\mathbf{q}_1$ and $\mathbf{q}_2$ we get the condition $|d_i(\mathbf{q}_1) - d_i(\mathbf{q}_2)| \leq \| \mathbf{q}_1 - \mathbf{q}_2 \|$.

$D(\mathbf{q})$ is locally Lipschitz since a function expressed as the maximum (or minimum) of a finite number of locally Lipschitz functions is itself locally Lipschitz ([5] p. 47). ■

Proof of Lemma 0.5 Equation 3 is used to determine $\nabla \text{dist}(\mathbf{a}_t(\mathbf{q}), \mathcal{B}_i)$ by substituting $\mathbf{a}_i = \mathbf{a}_t(\mathbf{q})$ and $\mathbf{b}_i = \mathbf{b}_t(\mathbf{q})$. Without l.o.g apply a change of coordinates to the world frame $\mathcal{F}_{\mathcal{W}}$ such that one of the primary axis is parallel to $\mathbf{ab} = \mathbf{a}_i - \mathbf{b}_i$ and the other two are perpendicular, so that all but one of $\mathbf{ab}_x$, $\mathbf{ab}_y$ and $\mathbf{ab}_z$ are zero.

The x, y, z terms of Equation 3 (1^{st} row) are equal for all $\mathbf{a}_t(\mathbf{q})$, so certainly the finite subset $\{\mathbf{a}_t(\mathbf{q})\} \subset \overline{P}$ is sufficient to determine the first three terms of $\partial d_i(\mathbf{q})$. The ϕ term (2^{nd} row) is equal to $U_x \mathbf{ab}_y - U_y \mathbf{ab}_x$ and is illustrated with Figure 11. Without l.o.g. assume $\mathbf{ab}_x = 0$, and since $\mathbf{ab}_y$ is a constant we need only address the term $U_x = \mathbf{a}_t(\mathbf{q}) - [x, y, z]^T$. It is easily seen that U_x achieves a minimum at support point $\mathbf{a}_{t_1}(\mathbf{q})$ and maximum at the support point $\mathbf{a}_{t_2}(\mathbf{q})$ such that $U_x^{t_1} \leq U_x \leq U_x^{t_2}$. Thus it is sufficient to consider only the support points for the fourth term. The θ, ψ terms are handled similarly. Since the change of coordinates of $\mathcal{F}_{\mathcal{W}}$ is arbitrary, the lemma follows. ■

Proof of Lemma 0.6: We have the following for the pointwise maxima function $\phi_{max}(\mathbf{x}) = \max_i\{f_i(\mathbf{x})\}$ (Proposition 2.3.12 [5])

$$\partial \phi_{max}(\mathbf{x}) \subset \text{co}\{\partial f_i(\mathbf{x}) : i \in I(\mathbf{x})\}$$

Since $\min_i\{f_i(\mathbf{x})\} = -\max_i\{-f_i(\mathbf{x})\}$, so we have

$$\begin{aligned} \partial \phi_{min}(\mathbf{x}) &= \partial(-\max_i\{-f_i(\mathbf{x})\}) \\ &= -\partial(\max_i\{-f_i(\mathbf{x})\}) \text{ (Prop. 2.3.1 [5])} \\ &\subset -\text{co}\{-\partial f_i(\mathbf{x}) : i \in I(\mathbf{x})\} \\ &= \text{co}\{\partial f_i(\mathbf{x}) : i \in I(\mathbf{x})\} \end{aligned}$$ ■

Proof of Lemma 0.7: If $f_i(\mathbf{x})$ is smooth at $\mathbf{x}$, then by Equation 5, $f_i^\circ(\mathbf{x}; \mathbf{e}) = \langle \nabla f_i(\mathbf{x}), \mathbf{e} \rangle$.

The case where $\mathbf{e} \in -E_i$ is trivial. For the case, $\mathbf{e} \notin E, -E_i$, assume no sequence $\{\mathbf{x}_k\}$ lying in $\mathbf{x} + E_i$ exists such that $\limsup_{k \to \infty} f_i^\circ(\mathbf{x}_k; \mathbf{e}) \geq 0$. We then know that $f_i^\circ(\mathbf{x}_k; \mathbf{e}) = \langle \nabla f_i(\mathbf{x}_k), \mathbf{e} \rangle < 0$ for all $\mathbf{x}_k \in \mathbf{x} + E_i$ near $\mathbf{x}$. Now, there exists a direction $\mathbf{v} \in \text{bndy}(\text{cl}(E_i))$ such that $\langle \mathbf{v}, \mathbf{e} \rangle < 0$, so $\langle \nabla f_i(\mathbf{x}_k), \mathbf{v} \rangle > 0$. We then have $f_i^\circ(\mathbf{x}; \mathbf{v}) \geq \limsup_{k \to \infty} f_i^\circ(\mathbf{x}_k; \mathbf{v}) > 0$, which contradicts that fact that $f_i^\circ(\mathbf{x}; \mathbf{v}) = 0$. ■

Proof of Lemma 0.9: Recall that $\mathbf{0} \in \partial f(\mathbf{q})$ ($\mathbf{q}$ is critical) iff $f^\circ(\mathbf{q}; \mathbf{e}) \geq 0$ for all $\mathbf{e}$. If $\mathbf{q}$ is regular, there exists a direction $\mathbf{e}$ such that $f^\circ(\mathbf{q}; \mathbf{e}) < 0$ and $f^\circ(\mathbf{q}; -\mathbf{e}) > 0$. By definition of f°

$$(d_i - d_j)^\circ(\mathbf{q}; \mathbf{e}) \leq d_i^\circ(\mathbf{q}; \mathbf{e}) + d_j^\circ(\mathbf{q}; -\mathbf{e}) \tag{8}$$

$$(d_i - d_j)^\circ(\mathbf{q}; -\mathbf{e}) \geq d_j^\circ(\mathbf{q}; \mathbf{e}) - d_i^\circ(\mathbf{q}; \mathbf{e}) \tag{9}$$

Since $\partial d_i(\mathbf{q}) \cap \partial d_j(\mathbf{q}) = \varnothing$, and are compact [5], there exist points $\boldsymbol{\zeta}_i \in \partial d_i(\mathbf{q})$ and $\boldsymbol{\zeta}_j \in \partial d_j(\mathbf{q})$ that attain the minimum distance between $\partial d_i(\mathbf{q})$ and $\partial d_j(\mathbf{q})$. Let $\mathbf{e} = \boldsymbol{\zeta}_j - \boldsymbol{\zeta}_i$, so by the Separation Theorem $\langle \boldsymbol{\zeta}_i, \mathbf{e} \rangle < \left\langle \boldsymbol{\zeta}_j, \mathbf{e} \right\rangle$ and

$$\begin{aligned} d_i^\circ(\mathbf{q}; \mathbf{e}) &= \langle \boldsymbol{\zeta}_i, \mathbf{e} \rangle < \left\langle \boldsymbol{\zeta}_j, \mathbf{e} \right\rangle \\ d_j^\circ(\mathbf{q}; -\mathbf{e}) &= -\left\langle \boldsymbol{\zeta}_j, \mathbf{e} \right\rangle \end{aligned}$$

so the RHS of Equation 8 is negative and the LHS of Equation 9 is positive, so $(d_i - d_j)(\mathbf{q})$ is regular at $\mathbf{q}$. ■