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TOPOLOGICAL GAMES IN DOMAIN THEORY

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Abstract

We prove that a metric space may be realized as the set of maximal elements in a continuous dcpo if and only if it is completely metrizable by showing more generally that the space of maximal elements in a domain is always complete in a sense first introduced by Choquet.

1 Introduction

Interest in the space of maximal elements in a domain can be traced at least as far back as twenty years ago, to the work of Scott [30], Kamimura and Tang [15], and Abramsky [1]. In the early 1990's, Edalat demonstrated that a fair amount of classical mathematics takes place at the top of a domain, through a series of papers detailing satisfying connections between domain theory and fundamental themes in analysis (integration [7], measure theory [9] and dynamical systems [8]). At the heart of such applications is that certain metric spaces X have domain theoretic *models,* i.e., there is a continuous dcpo $\mathbf{M}X$ such that $X \simeq \max(\mathbf{M}X)$, where the maximal elements $\max(\mathbf{M}X)$ are regarded a space in their relative Scott topology.

One use for a domain theoretic model of a space is that it can serve as a data type for describing the computation of elements in the underlying space; this direction has been pursued with vigor by Edalat and his colleagues [10]. Another, of a decidely different character, is the ability of a model to provide a setting in which the theorems of mathematics may be improved upon. For example, the order theoretic structure present in a model of a space can be used to establish new fixed point theorems, or to formulate the analytic notion derivative [21]. A crucial step in determining the applicability of results about models is obtaining a description of the spaces they capture as the set of maximal elements in the more familiar and traditional language of topology.

Casting the pragmatic aside, the question "Which classical spaces have domain theoretic models?" stands on its own as an alluring foundational issue in need of resolution. It is for this reason that interest in *the model problem,* as it is known in certain circles, has been steadily growing since the early 1990's, as evidenced for example by the papers of Lawson([18][19]), Flagg and Kopperman([5][12]), Alessi, Baldan and Honsell [3] and Mislove [28].

In this paper we offer significant progress on the model problem. The two main contributions of the present work are

- The solution of the model problem in domain theory for metric spaces, and more generally,
- The confirmation that the space of maximal elements in a domain is topologically complete.

In the process, a solid step toward the reconciliation of domains and the more traditional spaces of mathematics is taken, as asked for in Abramsky and Jung's [2] elegant account of the subject.

This paper begins by recalling some background definitions and the major achievements in work done on the model problem. The proof our main result uses a technique popular in classical descriptive set theory [17] that was introduced by Choquet [4] more than thirty years ago, a certain "topological game." It is a remarkably important idea that few people are familiar with. For this reason, we will devote some time to discussing the nuances of topological games before proceeding to the new results.

2 Domain theory

Let $(P, \sqsubseteq)$ be a partially ordered set or *poset* [2]. A nonempty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* $\bigsqcup S$ of $S \subseteq P$ is the least of its upper bounds when it exists. A *dcpo* is a poset in which every directed set has a supremum.

For elements x, y of a dcpo D, we write $x \ll y$ iff for every directed subset S with $y \sqsubseteq \bigsqcup S$, we have $x \sqsubseteq s$, for some $s \in S$. In the special case that this occurs for $x = y$, we call x *compact.*

Definition 2.1 Let $(D, \sqsubseteq)$ be a dcpo. We set

- $\twoheaddownarrow x := \{y \in D : y \ll x\}$ and $\twoheaduparrow x := \{y \in D : x \ll y\}$
- $\downarrow x := \{y \in D : y \sqsubseteq x\}$ and $\uparrow x := \{y \in D : x \sqsubseteq y\}$

and say D is *continuous* if $\twoheaddownarrow x$ is directed with supremum x for each $x \in D$. A *domain* is a continuous dcpo.

The *Scott topology* on a domain D has as a basis all sets of the form $\twoheaduparrow x$ for $x \in D$.

Definition 2.2 A subset B of a dcpo D is a *basis* for D if $B \cap \twoheaddownarrow x$ contains a directed subset with supremum x, for each $x \in D$. A dcpo is *algebraic* if its compact elements form a basis. A dcpo is *ω-continuous* if it has a countable basis.

Definition 2.3 A *Scott domain* is a continuous dcpo with a least element in which every pair of elements bounded from above has a supremum.

3 Models of spaces

Before proceeding, let us remind the reader that a space is *metrizable* if its topology can be realized as the metric topology with respect to some metric; *completely metrizable* when its topology is given by a metric all of whose Cauchy sequences converge; *Polish* when it is completely metrizable and has a countable base; and *zero-dimensional* when it has a base consisting of sets which are both open and closed.

Definition 3.1 A *model* of a space X is a continuous dcpo $\mathbf{M}X$ together with a homeomorphism

$$\phi : X \to \max(\mathbf{M}X)$$

where $\max(\mathbf{M}X)$ carries its relative Scott topology inherited from $\mathbf{M}X$. If in addition the domain $\mathbf{M}X$ is ω-continuous, then $(\mathbf{M}X, \phi : X \simeq \max(\mathbf{M}X))$ is called a *countably based model.*

The first result combines work in papers written thirteen years apart; one direction was provided by Kamimura and Tang [15], the other by Flagg and Kopperman [12].

Theorem 3.1 *A space has a model by an ω-algebraic Scott domain iff it is Polish and zero-dimensional.*

The solution for continuous Scott domains is also the result of two separate efforts: Lawson [18] and then Flagg, Kopperman and Ciesielski [5].

Theorem 3.2 *A space has a model by an ω-continuous Scott domain iff it is Polish.*

The solution for countably based models of metric spaces is also due to two different authors: Lawson [18], who proved that all Polish spaces have countably based models, and the present author, who provided the converse [27].

Theorem 3.3 *A regular space has a countably based model iff it is Polish.*

In all the results above, there are two directions that must be established, and neither ought to be underestimated: Have faith that there is a good reason why every solution to the model problem requires results from at least two different papers! Amusingly, this tradition will be honored in the present paper, using the first known model of complete metric spaces, discovered by Edalat and Heckmann [6]:

Example 3.1 For a complete metric space (X, d),

$$\mathbf{B}X = X \times [0, \infty)$$

ordered by

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s.$$

is a continuous dcpo with

$$(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s.$$

If A is a dense subset of X, then $A \times \mathbb{Q}$ is basis for $\mathbf{B}X$. The domain $\mathbf{B}X$ is a model of X because $X \simeq \max \mathbf{B}X = \{(x, 0) : x \in X\}$.

Notice the way complete metrizability appears on the surface as indispensable: An increasing sequence leads to a Cauchy sequence, which has a supremum iff it has a limit in the metric topology. All attempts to model metric spaces have essentially involved the same idea [27]. The question of the hour: Is complete metrizability necessary in order to construct a model of a metric space?

4 A topological game

We now turn attention to the valuable idea that will be applied to establish the main result of this paper. In this topological *game* there are two players named α and β. Play begins with player β, who produces an element x_1 from some fixed collection of plays D. Player α counters this by playing some $x_2 \in D$. This activity continues and from it emerges a sequence (x_n) in D, the x_i with i odd are plays made by β, those with i even are made by α. Such a sequence is commonly referred to as a *run* of the game.

Like all games, there are *rules* that each player must follow in making a play, i.e., they are free to do as they like provided that they do so *within the constraints imposed by the game.* These rules are an important component in distinguishing one game from another.

In the game introduced by Choquet, usually called the "strong Choquet" game [17], one uses a topological space (X, τ) to define the legal plays

$$D = \{(U, x) \in \tau \times X : x \in U\}.$$

The rules of the game, which define how each player is allowed to play, are specified as follows: If β has played $(U, x) \in D$, then α *must play* an element $(V, x) \in D$ with $x \in V \subseteq U$; If α has just played $(V, x) \in D$, then β *must play* any $(W, y) \in D$ with $y \in W \subseteq V$. There is an important distinction between the two. Essentially, the odds are stacked against player α: He is not allowed to change points, while player β is. Thus, a run of our game results in a sequence $(V_n, x_n)_{n \geq 1}$ in D such that $V_{n+1} \subseteq V_n$ for each $n \geq 1$ and $x_{n+1} = x_n$ for each odd $n \geq 1$.

Finally, we come to the crucial point about this game: What it means for a player to *win.* For this, we first decide what it means for a player to win a *particular* run, and then say that a player wins the game if he wins *every* run of the game. In the Choquet game, we say that player α wins a run (V_n, x_n) of the game if $\bigcap V_n \neq \emptyset$. This brings us to the notion of a strategy.

Intuitively, a strategy gives α a systematic method for playing the game. Formally, a *strategy* for α in the Choquet game is a function

$$\sigma : \bigcup_{n \geq 1} D^n \to D$$

with $(\pi_1 \circ \sigma)(p_1, \ldots, p_n) \subseteq \pi_1(p_n)$ and $(\pi_2 \circ \sigma)(p_1, \ldots, p_n) = \pi_2(p_n)$, where $\pi_1 : D \to \tau$ and $\pi_2 : D \to X$ are the natural projection maps.

The reason strategies are defined on *all finite sequences* from D, and not only on D itself, is so that α is allowed to make use of all available information during a particular run of the game. That is, during a particular run, player α's next move can be based on *all* of β's previous moves $(p_1, \ldots, p_n)$, and not merely on β's most recent move p_n.

We say that α has a *winning strategy* if there is a strategy σ that player α can follow which enables him to win every run of the game: That is, there exists a strategy σ for α such that for every sequence of plays (p_n) in D with $\pi_1(p_{n+1}) \subseteq (\pi_1 \circ \sigma)(p_1, \ldots, p_n)$, we have $\bigcap_{n \geq 1} \pi_1(p_n) \neq \emptyset$. Intuitively, the (p_n) are plays made by β.

Definition 4.1 A topological space is *Choquet complete* if player α has a winning strategy in the Choquet game (defined above).

To some extent, this notion allows for the topological reconciliation of domains with the spaces encountered in classical mathematics, as asked for in Abramsky and Jung [2].

Theorem 4.1 *We have the following standard facts:*

(i) *A Choquet complete space is Baire.*

(ii) *A locally compact sober space is Choquet complete.*

(iii) *A metric space is Choquet complete iff it is completely metrizable.*

A proof of (ii) appears in [24]. In particular, a domain in its Scott topology is Choquet complete, as is a locally compact Hausdorff space. The others are all due to Choquet [4]. The only fact we make use of here is (iii).

5 The completeness of the maximal elements

We now come to the main result of this paper: Any space with a model is Choquet complete. The idea of the proof is simple: We prove the existence of a strategy which, if followed by player α, implies that every run of the Choquet game at the *top* results in an increasing sequence *within the domain.*

It comes as something of a surprise that such a construction is actually possible, and what it turns on is that a strategy for player α is allowed to base his next move on *all* of player β's previous moves, and not just on player $\beta's$ most recent move. This freedom in the notion of strategy, in conjunction with both directed completeness as well as the existence of approximations at the top, makes the following theorem possible.

Theorem 5.1 *A space with a model is Choquet complete.*

Proof. Let X be the set of maximal elements in a model $\mathbf{M}X$ and let D be the set of plays in the Choquet game on X in its relative Scott topology. Thus, the members of D are of the form $(U \cap X, x)$, where U is a Scott open subset of $\mathbf{M}X$.

For the proof, we are going to define a winning strategy σ for player α by induction on the length n of sequences in D^n. First we define σ on $D = D^1$. Given $(U_1 \cap X, x_1) \in D$, we know that $x_1 \in U_1$, so by the continuity of $\mathbf{M}X$ at x_1, there is $b_1 \ll x_1$ with $b_1 \in U_1$. Then set $\sigma(U_1 \cap X, x_1) = (\Uparrow b_1 \cap X, x_1)$.

Now assume that σ is defined on $\bigcup_{1 \leq i \leq n} D^i$ for some $n \geq 1$. To define σ at $((U_1 \cap X, x_1), \ldots, (U_{n+1}, x_{n+1})) \in D^{n+1}$, we consider two cases:

(i) $(\forall i \in \{1, \ldots, n\})\, U_{i+1} \cap X \subseteq (\pi_1 \circ \sigma)((U_1 \cap X, x_1), \ldots, (U_i \cap X, x_i))$

(ii) $(\exists i \in \{1, \ldots, n\})\, U_{i+1} \cap X \not\subseteq (\pi_1 \circ \sigma)((U_1 \cap X, x_1), \ldots, (U_i \cap X, x_i))$

The case of (ii) does not correspond to a run of the game and is essentially irrelevant, so we define σ in this instance as:

$$\sigma((U_1 \cap X, x_1), \ldots, (U_{n+1} \cap X, x_{n+1})) = (U_{n+1} \cap X, x_{n+1}).$$

In the case of (i), a run of the game is underway in which both players have made n plays and player β has just made its $(n+1)^{st}$ move. Player α should follow with

$$\sigma((U_1 \cap X, x_1), \ldots, (U_{n+1} \cap X, x_{n+1})) = (\twoheaduparrow b_{n+1} \cap X, x_{n+1}),$$

where b_{n+1} is an approximation of x_{n+1} belonging to the Scott open set $U_{n+1} \cap \twoheaduparrow b_n$ and $\sigma((U_1 \cap X, x_1), \ldots, (U_n \cap X, x_n)) = (\twoheaduparrow b_n \cap X, x_n)$.

Player α can now win every run of the game by playing according to the strategy σ: Any sequence $(U_n \cap X, x_n)$ in D with

$$U_{n+1} \cap X \subseteq (\pi_1 \circ \sigma)((U_1 \cap X, x_1), \ldots, (U_n \cap X, x_n))$$

for all $n \geq 1$ yields a sequence (b_n) in $\mathbf{M}X$ such that $b_n \ll b_{n+1}$ and $b_n \in U_n$ for all $n \geq 1$. Because $\mathbf{M}X$ is a dcpo, we know that $b := \bigsqcup b_n$ exists. Because Scott open sets are upper, $b \in U_n$ for each n, leaving

$$\emptyset \neq \uparrow b \cap X \subseteq \bigcap_{n \geq 1} (U_n \cap X),$$

where $\uparrow b \cap X$ is nonempty because $\mathbf{M}X$ is a dcpo (Hausdorff maximality). $\square$

This gives a second proof of the Baire theorem [24],

Corollary 5.1 *The space of maximal elements in a domain is Baire.*

and also extends our results on models of metric spaces to final form:

Corollary 5.2 *The space of maximal elements in a domain is metrizable iff completely metrizable.*

Finally, combining the present effort with that of Abbas Edalat and Reinhold Heckmann [6] in Example 3.1 yields the solution to the model problem for metric spaces:

Corollary 5.3 *A metric space has a model iff it is completely metrizable.*

We can now *define* a metric space to be completely metrizable just when it is homeomorphic to $\max(D)$ for some domain D.

6 Stationary strategies on compact domains

What makes the application of Choquet completeness to the space of maximal elements in a domain so natural is the freedom it permits in the notion of strategy: In a given run, player α is allowed to base his next move on *all* of the previous moves made by β.

But in a great number of cases, β is not a very formidable opponent: Instead of having to examine all previous β moves, player α can usually win by considering only the most recent β move, ignoring all others. To formalize this, we write last : $\bigcup_{n\geq 1} D^n \to D$ for the map

$$\mathrm{last}(p_1, \ldots, p_n) = p_n$$

and call a strategy $\sigma : \bigcup_{n\geq 1} D^n \to D$ *stationary* when $\sigma = \sigma \circ \mathrm{last}$. Thus, a stationary winning strategy for α in the Choquet game is a map $\sigma : D \to D$ such that

(i) For any $p \in D$, $\pi_1(\sigma p) \subseteq \pi_1(p)$ and $\pi_2(\sigma p) = \pi_2(p)$, and

(ii) For any (p_n) in D with $\pi_1(p_{n+1}) \subseteq \pi_1(\sigma p_n)$, we have $\bigcap \pi_1(p_n) \neq \emptyset$.

Čech-complete spaces provide well-known examples of spaces where player α can win the Choquet game by playing a stationary strategy [29]; Domains in their Scott topology are another [24].

Theorem 6.1 *Let* $\mathbf{M}X$ *be a Scott domain. Then player* α *has a stationary winning strategy in the Choquet game on* $\max(\mathbf{M}X)$.

Proof. As before, let X be the set of maximal elements in $\mathbf{M}X$ and D be the set of plays in the Choquet game on X in its relative Scott topology.

To define a stationary winning strategy $\sigma : D \to D$ for player α, given $(U \cap X, x) \in D$, we know that $x \in U$, so by the continuity of $\mathbf{M}X$ at x, there are $a \ll b \ll x$ with $a \in U$. Then set $\sigma(U \cap X, x) = (\twoheaduparrow b \cap X, x)$.

Any run of the game gives rise to a sequence $(U_n \cap X, x_n) \in D$ with $U_{n+1} \cap X \subseteq (\pi_1 \circ \sigma)(U_n \cap X, x_n)$. We have

$$x_{n+1} \in U_{n+1} \cap X \subseteq (\pi_1 \circ \sigma)(U_n \cap X, x_n) = \twoheaduparrow b_n \cap X \subseteq \uparrow a_n \cap X \subseteq U_n \cap X,$$

which shows that $a_i \sqsubseteq x_{n+1}$ for $1 \leq i \leq n$. Because $\mathbf{M}X$ is a Scott domain, the supremum $y_n = \bigsqcup_{i=1}^{n} a_i$ exists.

The sequence (y_n) is increasing, so by directed completeness of $\mathbf{M}X$, $\bigsqcup y_n$ exists and belongs to each $\uparrow a_n \subseteq U_n$. By Hausdorff maximality, there is an $m \in X$ with $\bigsqcup y_n \sqsubseteq m$. Invariably,

$$m \in \bigcap (U_n \cap X) \neq \emptyset,$$

which proves that σ is a stationary winning strategy for α. $\square$

In the presence of developability, the space of maximal elements in a Scott domain is also Čech-complete [26].

7 Ideas

Some unavoidable questions:

(i) Is there a space with a countably based model in which player α cannot win with a stationary strategy? Can such a space be developable?

(ii) If $\max(\mathbf{M}X)$ is developable, then Theorem 5.1 and Theorem 6.1 remain valid for *closed subsets* of $\max(\mathbf{M}X)$. Do these theorems *always* hold for closed subsets of $\max(\mathbf{M}X)$?

(iii) Is there a notion of topological completeness that applies to spaces with no separation which is hereditary for closed and G_δ sets, equivalent to complete metrizability for metric spaces, and also satisfied by locally compact sober spaces?

8 Closing

An abstract view of the model problem for metric spaces may help the reader appreciate the genuine value within domain theory of the topological game due to Choquet: We have a metrizable subset Y of a larger space X and seek to deduce that Y is completely metrizable. The set Y cannot be assumed closed, nor G_δ, and the space X has at most T_0 separation.

Choquet completeness is the only technique we have ever encountered capable of handling such a general situation. A proper strengthening of it could provide mathematics with a notion of topological completeness that applies in *all* disciplines.

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