Programming Research Group

THE COMPLEXITY OF PARTITION FUNCTIONS

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Abstract

We give a complexity theoretic classification of the counting versions of so-called H-colouring problems for graphs H that may have multiple edges between the same pair of vertices. More generally, we study the problem of computing a weighted sum of homomorphisms to a weighted graph H.

The problem has two interesting alternative formulations: First, it is equivalent to computing the partition function of a spin system as studied in statistical physics. And second, it is equivalent to counting the solutions to a constraint satisfaction problem whose constraint language consists of two equivalence relations.

In a nutshell, our result says that the problem is in polynomial time if the adjacency matrix of H has row rank 1, and #P-complete otherwise.

1 Introduction

This paper has two different motivations: The first is concerned with constraint satisfaction problems, the second with "spin-systems" as studied in statistical physics. A known link between the two are so-called H-colouring problems. Our main result is a complete complexity theoretic classification of the problem of counting the number of solutions of an H-colouring problem for an undirected graph H which may have multiple edges, and actually of a natural generalisation of this problem to weighted graphs H. Translated to the world of constraint satisfaction problems, this yields a classification of the problem of counting the solutions to constraint satisfaction problems for two equivalence relations. Translated to the world of statistical physics, it gives a classification of the problem of computing the partition function of a spin system.

Let us describe our result from each of the different perspectives: Let H be a graph, possibly with multiple edges between the same pair of vertices. An *H-colouring* of a graph G is a homomorphism from G to H. Both the decision problem, asking whether a given graph has an H-colouring, and the problem of counting the H-colourings of a given graph, have received considerable attention [5, 6, 9, 11, 12]. Here we are interested in the counting problem. Dyer and Greenhill [5] gave a complete complexity theoretic classification of the counting problem for undirected graphs H without multiple edges; they showed that the problem is in polynomial time if each connected component of H is complete bipartite without any loops or is complete with all loops present, and #P-complete otherwise. Here we are interested in counting H-colourings for graphs H that may have multiple edges. Note that, as opposed to the decision problem, multiple edges do make a difference for the counting problem. Let H be a graph with vertex set $\{1,\ldots,k\}$. H is best described in terms of its adjacency matrix $A=(A_{ij})$, where A_{ij} is the number of edges between vertices i and j. Given a graph $G=(V,E)$, we want to compute the number of homomorphisms from G to H. Observe that this number is

$$Z_A(G) = \sum_{\sigma:V\to\{1,\ldots,k\}} \prod_{e=\{u,v\}\in E} A_{\sigma(u)\sigma(v)}. \tag{1}$$

Borrowing from the physics terminology, we call Z_A the *partition function* of A (or H). We denote the problem of computing $Z_A(G)$ for a given graph G by $\mathrm{EVAL}(A)$. Of course if we define Z_A as in (1), the problem is not only meaningful for matrices A that are adjacency matrices of graphs, but for arbitrary square matrices A. We may view such matrices as adjacency matrices of weighted graphs (omitting edges of weight 0). We call a symmetric matrix A *connected* (*bipartite*) if the corresponding graph is connected (*bipartite*, respectively).

We prove the following classification result:

Theorem 1 *Let A be a symmetric matrix with non-negative real entries.*

(1) If A is connected and not bipartite, then $\mathrm{EVAL}(A)$ *is in polynomial time if the row rank of A is at most 1; otherwise* $\mathrm{EVAL}(A)$ *is #P-complete.*

(2) If A is connected and bipartite, then EVAL(A) *is in polynomial time if the row rank of A is at most 2; otherwise* EVAL(A) *is #P-complete.*

(3) If A is not connected, then EVAL(A) *is in polynomial time if each of its connected components satisfies the corresponding condition stated in (1) or (2); otherwise* EVAL(A) *is #P-complete.*

Note that this generalises Dyer and Greenhill's [5] classification result for graphs without multiple edges, whose adjacency matrices are symmetric 0-1 matrices.

Our proof builds on interpolation techniques similar to those used by Dyer and Greenhill, recent results on counting the number of solutions to constraint satisfaction problems due to Dalmau and the first author [1], and a considerable amount of polynomial arithmetic. Even though we present the proof in the language of constraint satisfaction problems here, in finding the proof it has been very useful to jump back and forth between the H-colouring and constraint satisfaction perspective.

Let us now explain the result for constraint satisfaction problems. A *constraint language* Γ on a finite *domain* D is a set of relations on D. An instance of the problem CSP(Γ) is a triple $(V, D, \mathcal{C})$ consisting of a set V of *variables*, the domain D, and a set $\mathcal{C}$ of *constraints* $\langle s, \rho \rangle$, where, for some $r \geq 1$, $s \in V^r$ and ρ is an r-ary relation in Γ. A *solution* is a mapping $\sigma : V \to D$ such that for each constraint $\langle (v_1, \dots, v_r), \rho \rangle \in \mathcal{C}$ we have $(\sigma(v_1), \dots, \sigma(v_r)) \in \rho$. There has been considerable interest in the complexity of constraint satisfaction problems [16, 14, 7, 2, 3], which has mainly been driven by Feder and Vardi's [7] *dichotomy question*, asking whether for all languages Γ the problem CSP(Γ) is either solvable in polynomial time or NP-complete. A similar dichotomy question can be asked for the problem #CSP(Γ) of counting the solutions for a given instance [4, 1].

We consider constraint languages Γ consisting of two equivalence relations α, β. Suppose that α has k equivalence classes and β has ℓ equivalence classes. Then Γ can be described by a $(k \times \ell)$-matrix $B = (B_{ij})$, where B_{ij} is the number of elements in the intersection of the ith class of α and the jth class of β. We show that, provided that the matrix is "indecomposable" (in a sense made precise in Section 2.1), the problem #CSP(Γ) is in polynomial time if the row rank of B is 1 and #P-complete otherwise. There is also a straightforward extension to "decomposable" matrices. In [1], it has been shown that if #CSP(Γ) is in polynomial time, then Γ has a so-called *Mal'tsev polymorphism.* The result of this paper provides a further necessary condition for Γ to give rise to a counting problem solvable in polynomial time.

We can generalise the result to *weighted CSP*, where each domain element d carries a non-negative real weight $\omega(d)$. The weight of a solution $\sigma : V \to D$ is defined to be the product $\prod_{v \in V} \omega(\sigma(v))$, and the goal is to compute the weighted sum over all solutions. As an important intermediate step, we even prove our classification result for weights that are polynomials with integer coefficients.

Let us finally describe the connection with statistical physics. Statistical physics explains properties of substances, such as gases, liquids or crystals, using probability distributions on certain states of the substance. In one of the standard models, a substance is considered as a conglomeration of particles (atoms) viewed as a graph $G = (V, E)$, also called a *lattice*, in which adjacent vertices represent particles interacting in a non-negligible way. Every particle may have one of k *spins*; the interaction between neighbouring particles can be described by a *spin system*, which is just a symmetric $k \times k$-matrix $K = (K_{ij})$. The entry K_{ij} of K corresponds, in a certain way, to the energy that a pair of interacting particles, one of which has spin i, the other one has spin j, contributes into the overall energy of G. A *configuration* of the system on a graph $G = (V, E)$ is a mapping $\sigma : V \to \{1, \ldots, k\}$. The *energy* of σ is the sum $H(\sigma) = \sum_{e=\{u,v\}\in E} K_{\sigma(u)\sigma(v)}$. Then the probability that G has configuration σ is $\frac{1}{Z}\exp(-H(\sigma)/cT)$, where $Z = \sum_\sigma \exp(-H(\sigma)/cT)$ is the *partition function* and T is a parameter of the system (the *temperature*) and c is a constant. As is easily seen, this probability distribution obeys the law "the lower energy a configuration has, the more likely it is". Observe that $Z = Z_A(G)$ for the matrix A with

$$A_{ij} = \exp(-K_{ij}/cT).$$

Thus EVAL(A) is just the problem of computing the partition function for the system described by A. Dyer and Greenhill in [5] dealt with spin systems in which certain configuration are prohibited and the others are uniformly distributed, while our results are applicable to arbitrary spin systems.

Preliminaries

$\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ denote the real numbers, rational numbers and integers, respectively, and $\mathbb{Q}[X]$ and $\mathbb{Z}[X]$ denote the polynomial rings over $\mathbb{Z}$ and $\mathbb{Q}$ in an indeterminate X. Throughout this paper, we let $\mathbb{S}$ denote one of these five rings.

The *degree* of a polynomial $p(X)$ is denoted by $\mathsf{deg}(p)$.

For every set S, $S^{m\times n}$ denotes the set of all $m \times n$-matrices with entries from S. For a matrix A, A_{ij} denotes the entry in row i and column j. The row rank of a matrix $A \in \mathbb{S}^{m\times n}$ is denoted by $\mathsf{rank}(A)$. A matrix $A \in \mathbb{S}^{m\times n}$ is *non-negative* (*positive*), if, for $1 \le i \le m, 1 \le j \le n$, the leading coefficient of A_{ij} is non-negative (positive, respectively).

Graphs are always undirected, unless we explicitly call them *directed graphs*. Graphs and directed graphs may have loops and multiple edges. The *in-degree*, and *out-degree* of a vertex in a (directed) graph are defined in the obvious way and denoted by $\mathsf{indeg}(v)$, $\mathsf{outdeg}(v)$, respectively.

Our model of real number computation is a standard model, as it is, for example, underlying the complexity theoretic work on linear programming (cf. [10]). We can either assume that the numbers involved in our computations are polynomial time computable or that they are given by an oracle (see [15] for a detailed description of the model). However, our results do not seem to be very model dependent. All we really need is that the basic arithmetic operations are polynomial time computable. Our situation is fairly simple because all real numbers we encounter are the entries of some matrix A, which is always considered fixed, and numbers computed from the entries of A using a polynomial number of arithmetic operations. Instances of the problem EVAL(A) are just graphs, and we do not have to worry about real numbers as inputs of our computations.

2 The Tractable Cases

2.1 Block Decompositions

Let $B \in \mathbb{S}^{k\times\ell}$. A *submatrix* of B is a matrix obtained from B by deleting some rows and columns. For non-empty sets $I \subseteq \{1,\dots,k\}$, $J \subseteq \{1,\dots,\ell\}$, where $I = \{i_1,\dots,i_p\}$ with $i_1 < \dots < i_p$ and $J = \{j_1,\dots,j_q\}$ with $j_1 < \dots < j_q$, B_{IJ} denotes the $(p \times q)$-submatrix with $(B_{IJ})_{rs} = B_{i_r j_s}$ for $1 \le r \le p, 1 \le s \le q$. A *proper submatrix* of B is a submatrix $B' \neq B$.

Definition 2 *Let $B \in \mathbb{S}^{k\times\ell}$.*

(1) A decomposition *of B consists of two proper submatrices B_{IJ}, $B_{\overline{I}\,\overline{J}}$ such that*

(a) $\overline{I} = \{1,\dots,k\} \setminus I$,

(b) $\overline{J} = \{1,\dots,\ell\} \setminus J$,

(c) $B_{ij} = 0$ for all $(i,j) \in (I \times \overline{J}) \cup (\overline{I} \times J)$.

B is indecomposable *if it has no decomposition.*

(2) A block *of B is an indecomposable submatrix B_{IJ} with at least one non-zero entry such that $B_{IJ}, B_{\overline{I}\,\overline{J}}$ is a decomposition of B.*

Indecomposability may be viewed as a form of "connectedness" for arbitrary matrices. For square matrices there is also a natural graph based notion of connectedness.

Let $A \in \mathbb{S}^{k\times k}$ be a square matrix. A *principal submatrix* of A is a submatrix of the form A_{II} for some $I \subseteq \{1,\dots,k\}$. Instead of A_{II} we just write A_I. The *underlying graph* of A is the (undirected) graph $G(A)$ with vertex set $\{1,\dots,k\}$ and edge set $\{\{i,j\} \mid 1 \le i,j \le n \text{ such that } A_{ij} \neq 0 \text{ or } A_{ji} \neq 0\}$. Note that we define $G(A)$ to be an undirected graph even if A is not symmetric.

Definition 3 *Let $A \in \mathbb{S}^{k\times k}$.*

(1) A is connected *if the graph $G(A)$ is connected.*

(2) A connected component *of the matrix A is a principal submatrix A_C, where C is the vertex set of a connected component of $G(A)$.*

Lemma 4 *A connected symmetric matrix is either indecomposable or bipartite. In the latter case, the matrix has two blocks corresponding to the two parts of the bipartition.*

There is another useful connection between indecomposability and connectedness. For a matrix $B \in \mathbb{S}^{k\times\ell}$, let

$$\mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{S}^{(k+\ell)\times(k+\ell)}.$$

Note that $\mathsf{bip}(B)$ is the adjacency matrix of a weighted bipartite directed graph. The following lemma is straightforward.

Lemma 5 *Let $B \in \mathbb{S}^{k\times\ell}$ and $A = \mathsf{bip}(B)$. Then for every block B_{IJ} of B there is a connected component A_C of A such that $A_C = \mathsf{bip}(B_{IJ})$, and conversely for every connected component A_C of A there is a block B_{IJ} of B such that $A_C = \mathsf{bip}(B_{IJ})$.*

In particular, B is indecomposable if, and only if, A is connected.

2.2 Partition functions of graphs

Even though our main result is about symmetric matrices and (undirected) graphs, it is useful to generalise partition functions to directed graphs, which we do in the most straightforward way. Let $A \in \mathbb{S}^{k\times k}$ be a square matrix that is not necessarily symmetric and $G = (V, E)$ a directed graph. For every $\sigma : V \to \{1, \dots , k\}$ we let

$$\omega_A(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u)\sigma(v)},$$

and we let

$$Z_A(G) = \sum_{\sigma:V\to\{1,\dots,k\}} \omega_A(\sigma).$$

Note that if A is symmetric, $G = (V, E)$ a directed graph, and G_U the underlying undirected graph, then $Z_A(G_U) = Z_A(G)$. Thus by EVAL(A) we may denote the problem of computing $Z_A(G)$ for a given directed graph, with the understanding that for symmetric A we can always consider the input graph as undirected. We use $\mathsf{rank}(A)$ to denote the row rank of a matrix A.

Theorem 6 *(1) Let $A \in \mathbb{S}^{k\times k}$ be a matrix such that each connected component of A has row rank 1. Then EVAL(A) is in polynomial time.*

(2) Let $A \in \mathbb{S}^{k\times k}$ be a symmetric matrix such that each connected component of A either has row rank at most 1 or is bipartite and has row rank at most 2. Then EVAL(A) is in polynomial time.

Proof: (1) Let $A_1, \dots , A_\ell$ be the connected components of A. Then for every graph G with connected components $G_1, \dots , G_m$ we have

$$Z_A(G) = \prod_{i=1}^{m} \sum_{j=1}^{\ell} Z_{A_j}(G_i).$$

Thus without loss of generality we may assume that A is connected.

Then $\mathsf{rank}(A) \le 1$, and thus there are numbers $a_1, \dots , a_k, b_1, \dots , b_k \in \mathbb{R}$ such that for $1 \le i, j \le k$ we have:

$$A_{ij} = a_i \cdot b_j$$

(the b_j can be chosen to be the A_{1j} and $a_i = A_{i1}/A_{11}$). Let $G = (V, E)$ be a directed graph and $\sigma : V \to \{1, \dots, k\}$. Then

$$\omega_A(\sigma) = \prod_{(v,w)\in E} A_{\sigma(v)\sigma(w)} = \prod_{(v,w)\in E} a_{\sigma(v)} b_{\sigma(w)} = \prod_{v\in V} a_{\sigma(v)}^{\mathsf{outdeg}(v)} b_{\sigma(v)}^{\mathsf{indeg}(v)}.$$

Thus

$$Z_A(G) = \sum_{\sigma:V\to\{1,\dots,k\}} \omega_A(\sigma) = \sum_{\sigma}\prod_{v\in V} a_{\sigma(v)}^{\mathsf{outdeg}(v)} b_{\sigma(v)}^{\mathsf{indeg}(v)} = \prod_{v\in V}\sum_{i=1}^{k} a_i^{\mathsf{outdeg}(v)} b_i^{\mathsf{indeg}(v)}.$$

The last term can easily be evaluated in polynomial time.

(2) We may assume that A is connected and bipartite with $\mathsf{rank}(A) = 2$. Then there are $k_1, k_2 \geq 1$ such that $k_1 + k_2 = k$ and a matrix $B \in \mathbb{R}^{k_1\times k_2}$ with $\mathsf{rank}(B) = 1$ and

$$A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}.$$

Let $G = (V, E)$ be a graph. If G is not bipartitite then $Z_A(G) = 0$, therefore, we may assume that G is connected and bipartite, say, with bipartition V_1, V_2. Let G_{12} be the directed graph obtained from G by directing all edges from V_1 to V_2, and let G_{21} be the directed graph obtained from G by directing all edges from V_2 to V_1. Recall that

$$\mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{k\times k}.$$

We have

$$Z_A(G) = Z_{\mathsf{bip}(B)}(G_{12}) + Z_{\mathsf{bip}(B)}(G_{21}).$$

Since EVAL($\mathsf{bip}(B)$) is in polynomial time by Theorem 6(1), this shows that $Z_A(G)$ can be computed in polynomial time. ■

3 Constraint Satisfaction Problems

Recall that a *constraint language* Γ on a *domain* D is a set of relations on D. The pair (D, Γ) is occasionally called the *template* of the constraint satisfaction problem CSP(Γ). An *instance* of CSP(Γ) is a triple $(V, D, \mathcal{C})$ consisting of a set V of *variables*, the domain D, and a set $\mathcal{C}$ of *constraints* $\langle s, \rho\rangle$, where, for some $r \geq 1$, $s \in V^r$ and ρ is an r-ary relation in Γ. A *solution* is a mapping $\sigma : V \to D$ such that for each constraint $\langle (v_1, \dots, v_r), \rho\rangle \in \mathcal{C}$ we have $(\sigma(v_1), \dots, \sigma(v_r)) \in \rho$. #CSP($\Gamma$) is the problem of counting the number of solutions for a given instance $\mathcal{P}$. We shall now define a weighted version of this problem. Let Γ be a constraint language on a domain D and $\omega : D \to \mathbb{S}$ be a *weight* *function*. Slightly abusing notation, we also use ω to denote the weight of a solution $\sigma : V \to D$ for an instance $\mathcal{P} = (V, D, \mathcal{C})$ of CSP(Γ); we let

$$\omega(\sigma) = \prod_{v \in V} \omega(\sigma(v)).$$

We let

$$\mathcal{Z}_{\Gamma,\omega}(\mathcal{P}) := \sum_{\sigma} \omega(\sigma)$$

where the sum ranges over all solutions σ for $\mathcal{P}$. We denote the problem of computing $\mathcal{Z}_{\Gamma,\omega}$ by WCSP(Γ, ω). The triple (D, Γ, ω) is called the *weighted template* of the problem WCSP(Γ, ω)

3.1 CSPs with two Equivalence Relations

Our main results on (weighted) constraint satisfaction problems are concerned with constraint languages consisting of two equivalence relations, which we usually denote by α and β.

So suppose that α and β are equivalence relations on a domain D. Let $C_1, \dots, C_k$ be the equivalence classes of α and $D_1, \dots, D_\ell$ the equivalence classes of β. We define a matrix $B(\alpha, \beta) \in \mathbb{Z}^{k \times \ell}$ by

$$B(\alpha, \beta)_{ij} = |C_i \cap D_j|.$$

Conversely, for every non-negative integer matrix $B \in \mathbb{Z}^{k \times \ell}$ there are equivalence relations α_B, β_B on the domain

$$D_B = \{1, \dots, \sum_{\substack{1 \le i \le k \\ 1 \le j \le \ell}} B_{ij}\}$$

such that $B = B(\alpha_B, \beta_B)$. We fix such relations α_B, β_B and call $(D_B, \{\alpha_B, \beta_B\})$ the *canonical template* for B.

We extend the definitions to weighted problems. For a weight function $\omega : D \to \mathbb{S}$, we define a matrix $B(\alpha, \beta, \omega) \in \mathbb{S}^{k \times \ell}$ by

$$B(\alpha, \beta, \omega)_{ij} = \sum_{d \in C_i \cap D_j} \omega(d).$$

Conversely, for every matrix $B \in \mathbb{S}^{k \times \ell}$ we define a *canonical weighted template*

$$(D_B^w, \{\alpha_B^w, \beta_B^w\}, \omega_B^w)$$

as follows: The domain is $D_B^w = \{1, \dots, k\} \times \{1, \dots, \ell\}$, the equivalence relation α_B^w is equality on the first component, and β_B^w is equality on the second component. The weight function $\omega_B^w : D_B^w \to \mathbb{S}$ is defined by $\omega_B^w((i, j)) = B_{ij}$. Then clearly $B = B(\alpha_B^w, \beta_B^w, \omega_B^w)$.

The next lemma shows that the function $\mathcal{Z}_{\{\alpha,\beta\},\omega}$ only depends on the matrix $B(\alpha, \beta, \omega)$.

Lemma 7 *Let $(D, \{\alpha, \beta\}, \omega)$ and $(D', \{\alpha', \beta'\}, \omega')$ be two weighted templates, where α, β and α', β' are equivalence relations on D, D', respectively. Suppose that $B(\{\alpha, \beta\}, \omega) = B(\{\alpha', \beta'\}, \omega')$. Then $\mathcal{Z}_{\{\alpha,\beta\},\omega} = \mathcal{Z}_{\{\alpha',\beta'\},\omega}$.*

The proof is straightforward.

In the following, we write $\mathcal{Z}_B$ instead of $\mathcal{Z}_{\{\alpha_B^w,\beta_B^w\},\omega_B^w}$ and WCSP(B) instead of WCSP$(\{\alpha_B^w, \beta_B^w\}, \omega_B^w)$.

Corollary 8 *For every non-negative integer matrix $B \in \mathbb{Z}^{k\times\ell}$ the problems* #CSP$(\{\alpha_B, \beta_B\})$ *and* WCSP(B) *are equivalent (in the sense that each instance yields the same result).*

The following useful lemma is an immediate consequence of the definitions.

Lemma 9 *Let $B, B' \in \mathbb{S}^{k\times\ell}$ be such that B' is obtained from B by permuting rows and/or columns. Then $\mathcal{Z}_B = \mathcal{Z}_{B'}$.*

3.2 Back and Forth between CSP and H-colouring

The next lemma shows that weighted CSP for two equivalence relations are equivalent to evaluation problems for weighted bipartite graphs.

Lemma 10 *Let $B \in \mathbb{S}^{k\times\ell}$. Then the problems* WCSP(B) *and* EVAL$(\mathsf{bip}(B))$ *are polynomial time equivalent.*

Proof: Let

$$A = \mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{S}^{(k+\ell)\times(k+\ell)}.$$

Observe that for every directed graph $G = (V, E)$ we have $Z_A(G) = 0$ unless there is a bipartition V_1, V_2 of V such that $E \subseteq V_1 \times V_2$ (that is, all edges are directed from V_1 to V_2). In the following, we assume that there is such a bipartition V_1, V_2.

Let $(D, \{\alpha, \beta\}, \omega)$ be the canonical weighted template for B. We let $\mathcal{P}$ be the following instance of WCSP(B):

- The variables of $\mathcal{P}$ are the edges of G.
- The domain is $D = \{1, \ldots, k\} \times \{1, \ldots, \ell\}$, the domain of the canonical weighted template.
- For all edges $e, e' \in E$ that have the same endpoint in V_1, there is a constraint $\langle (e, e'), \alpha\rangle$.
- For all edges $e, e' \in E$ that have the same endpoint in V_2, there is a constraint $\langle (e, e'), \beta\rangle$.

We now show how to associate with every $\sigma : V \to \{1, \dots, k + \ell\}$ such that $\omega_A(\sigma) \neq 0$ a solution $\sigma^* : E \to D$ for $\mathcal{P}$.

Let $\sigma : V \to \{1, \dots, k + \ell\}$ such that $\omega_A(\sigma) \neq 0$. Then $\sigma(V_1) \subseteq \{1, \dots, k\}$ and $\sigma(V_2) \subseteq \{k+1, \dots, \ell\}$. We define $\sigma^* : E \to D$ by letting $\sigma^*((u, v)) = (\sigma(u), \sigma(v) - k)$ for every edge $(u, v) \in E$. It is not hard to see that σ^* is a solution for the instance $\mathcal{P}$. For example, if $e = (u, v), e' = (u, v') \in E$ are edges that have the same endpoint in V_1 then $\sigma^*(e)$ and $\sigma^*(e')$ have the same first coordinate $\sigma(u)$ and therefore are in relation α of the canonical weighted template. Thus the constraint $\langle (e, e'), \alpha \rangle$ is satisfied. Conversely, every solution for $\mathcal{P}$ is of the form σ^* for some σ with $\omega_A(\sigma) \neq 0$. Furthermore, we have

$$\omega_A(\sigma) = \prod_{(v,w) \in E} B_{\sigma(v)\sigma(w)} = \omega_B(\sigma^*).$$

Thus

$$Z_A(G) = \sum_{\sigma} \omega_A(\sigma) = \sum_{\sigma^*} \omega_B(\sigma^*) = \mathcal{Z}_B(\mathcal{P}).$$

This yields a reduction from EVAL(A) to WCSP(B).

For the converse direction, let $\mathcal{P} = (V, D, \mathcal{C}))$ be an instance of WCSP(B). Let

$$\alpha' := \{(u, v) \in V^2 \mid \langle (u, v), \alpha \rangle \in \mathcal{C}\} \text{ and } \beta' := \{(u, v) \in V^2 \mid \langle (u, v), \beta \rangle \in \mathcal{C}\}.$$

Without loss of generality we may assume that α' and β' are equivalence relations. To see this, just note that since α and β are equivalence relations, every solution $\sigma : V \to D$ also satisfies all constraints of the form $\langle (u, v), \alpha \rangle$, where (u, v) is in the reflexive symmetric transitive closure of α', and $\langle (u, v), \beta \rangle$, where (u, v) is in the reflexive symmetric transitive closure of β'. Let $C_1, \dots, C_k$ be the equivalence classes of α' and $D_1, \dots, D_\ell$ the equivalence classes of β'. Let G be the directed graph defined as follows: The vertex set is $\{1, \dots, k + \ell\}$, and for $1 \leq i \leq k$, $1 \leq j \leq \ell$ there are $|C_i \cap D_j|$ edges from i to $(k + j)$. It is easy to see that $\mathcal{Z}_B(\mathcal{P}) = Z_A(G)$. This yields a reduction from WCSP(B) to EVAL(A). ∎

The following lemma is needed to derive the hardness part of Theorem 1 from the hardness results on weighted CSP.

Lemma 11 *Let $A \in \mathbb{S}^{k \times k}$. Then* WCSP$(A)$ *is polynomial time reducible to* EVAL(A).

Proof: Let $A' = \mathsf{bip}(A)$. By Lemma 10, it suffices to prove that EVAL(A') is reducible to EVAL(A).

Let $G = (V, E)$ be a directed graph. If G is not bipartite with all edges directed from one part to the other, then $Z_{A'}(G) = 0$. Therefore, we assume that there is a partition V_1, V_2 of V such that $E \subseteq V_1 \times V_2$. We claim that

$$Z_{A'}(G) = Z_A(G). \tag{2}$$

Note that for every $\sigma' : V \to \{1, \dots, 2k\}$ with $\omega_{A'}(\sigma) \neq 0$ we have $\sigma(V_1) \subseteq \{1, \dots, k\}$ and $\sigma(V_2) \subseteq \{k+1, \dots, 2k\}$.

For $\sigma : V \to \{1, \dots, k\}$, let $f(\sigma) : V \to \{1, \dots, 2k\}$ be defined by $f(\sigma)(v_1) = \sigma(v_1)$ and $f(\sigma)(v_2) = \sigma(v_2) + k$ for all $v_1 \in V_1$, $v_2 \in V_2$. Then $\omega_A(\sigma) = \omega_{A'}(\sigma')$. Moreover, f is one-to-one, and for every $\sigma' : V \to \{1, \dots, 2k\}$ with $\omega_{A'}(\sigma') \neq 0$ there exists $\sigma : V \to \{1, \dots, k\}$ such that $\sigma' = f(\sigma)$. This proves (2). ∎

We close this section with another lemma which will be used later.

Lemma 12 *Let $B \in \mathbb{S}^{k \times \ell}$. Then* $\mathrm{EVAL}(B \cdot B^\top)$ *is polynomial time reducible to* $\mathrm{WCSP}(B)$.

Proof: By Lemma 10, it suffices to show that $\mathrm{EVAL}(B \cdot B^\top)$ is polynomial time reducible to $\mathrm{EVAL}(\mathsf{bip}(B))$.

For a given graph $G = (V, E)$, let $G' = (V', E')$ be the graph obtained from G by replacing every edge by two edges pointing to a new vertex. More precisely, let $V' = V \cup V_E$, where $V_E = \{v_e \mid e \in E\}$, and $E' = \{(u, v_{(u,v)}), (v, v_{(u,v)}) \mid (u, v) \in E\}$.

Observe that for every mapping $\sigma' : V' \to \{1, \dots, k + \ell\}$ such that $\omega_{\mathsf{bip}(B)}(\sigma') \neq \emptyset$ we have $\sigma(V) \subseteq \{1, \dots, k\}$ and $\sigma(V_E) \subseteq \{k+1, \dots, \ell\}$. Thus

$$
\begin{aligned}
Z_{\mathsf{bip}(B)}(G') &= \sum_{\sigma' : V' \to \{1, \dots, k+\ell\}} \omega_{\mathsf{bip}(B)}(\sigma') \\
&= \sum_{\sigma : V \to \{1, \dots, k\}} \sum_{\sigma_E : V_E \to \{k+1, \dots, \ell\}} \prod_{e = (u,v) \in E} B_{\sigma(u)\sigma_E(v_e) - k} B_{\sigma(v)\sigma_E(v_e) - k} \\
&= \sum_{\sigma : V \to \{1, \dots, k\}} \prod_{e = (u,v) \in E} \sum_{i=1}^{\ell} B_{\sigma(u)i} B_{\sigma(v)i} \\
&= Z_{B \cdot B^\top}(G).
\end{aligned}
$$

Thus the mapping $G \mapsto G'$ yields a polynomial time reduction from $\mathrm{EVAL}(B \cdot B^\top)$ to $\mathrm{EVAL}(\mathsf{bip}(B))$. ∎

4 The results

We are able now to state the main results of the paper in their most general form. The tractability part of these theorems follows from Theorem 6 and Lemmas 10, 11; the hardness part follows from Theorem 16, which ist stated and proved in the next section.

Theorem 13 *Let A be a symmetric matrix with non-negative entries from $\mathbb{S}$.* $\mathrm{EVAL}(A)$ *is in polynomial time if the row rank of each non-bipartite connected component of A is at most* 1, *and the row rank of each bipartite component is at most 2. Otherwise* $\mathrm{EVAL}(A)$ *is #P-complete.*

Theorem 14 *Let $B \in \mathbb{S}^{k \times \ell}$ be a non-negative matrix.* $\mathrm{WCSP}(B)$ *is in polynomial time if and only if the row rank of each block of B is at most 1. Otherwise* $\mathrm{WCSP}(B)$ *is #P-complete.*

Making use of Corollary 8 we derive a classification result for the counting constraint satisfaction problem.

Corollary 15 *Let α, β be equivalence relations on a set D. $\#\mathrm{CSP}(\{\alpha, \beta\})$ is in polynomial time if and only if the row rank of each block of $B(\alpha, \beta)$ is at most 1. Otherwise $\#\mathrm{CSP}(B)$ is #P-complete.*

5 The Main Hardness Theorem

Theorem 16 *Let $B \in \mathbb{S}^{k \times \ell}$ be non-negative such that at least one block of B has row rank at least 2. Then $\mathrm{WCSP}(B)$ is #P-complete.*

Outline of the proof. Before we prove Theorem 16, we give a brief outline of the proof. Let $B \in \mathbb{S}^{k \times \ell}$ be a non-negative matrix such that at least one block of B has row rank at least 2.

Step 1: From numbers to polynomials. In this first step of the proof we show that we can assume that all positive (i.e. non-zero) entries of B are powers of some indeterminate X. More precisely, we prove that there is a matrix B^* whose positive entries are powers of X such that B^* also has a block of row rank at least 2 and $\mathrm{WCSP}(B^*)$ is polynomial time reducible to $\mathrm{WCSP}(B)$. The construction is based on a lemma, which essentially goes back to [5], stating that the problem $\mathrm{WCSP}(B)$ is equivalent to the problem of counting all solutions of a given weight. For simplicity, let us assume here that all entries of B are non-negative integers; additional tricks are required for real matrices. We can use the lemma to filter out powers of a particular prime p from all entries of B. This way we obtain a matrix B' whose positive entries are powers of a prime p. Using a technique which corresponds to "thickening" in the graph context (cf. [13, 5]), we can replace the entries of this matrix by arbitrary powers, and by interpolation we can then replace p by the indeterminate X. This gives us the desired matrix B^*.

From now on, we assume that all positive entries of B are powers of X.

Step 2: Further preparations. Now we employ two results due to [1]. A consequence of the first is a lemma stating that if there are rows i, i' and columns j, j' such that the entries $B_{ij}, B_{ij'}, B_{i'j}$ are non-zero and $B_{i'j'}$ is zero, then $\mathrm{WCSP}(B)$ is #P-complete. This implies that we may assume that in every block of B all entries are positive.

The second result due to [1] is that the more general problem where we only count solutions in which the values of some variables are fixed (i.e., solutions extending some fixed partial solutions) is reducible to $\mathrm{WCSP}(B)$. This implies that we may assume that B is indecomposable.

Together, the two assumptions imply that all entries of B are positive.

Another simple reduction shows that either (a) each row of B and each column of B contains a 1, or (b) B has principal submatrix in which all entries are 1, and no 1s appear outside of this principal submatrix. From here we branch into cases (a) and (b).

Step 3(a): Separate 1s. We assume that all entries of B are positive and each row of B and each column of B contains a 1. Since we may permute rows and columns of B, we may assume that all diagonal entries of B are 1. It is not hard to see that then we can reduce the problem EVAL(A) for a symmetric non-singular 2×2-matrices to WCSP(B). For such matrices A the problem EVAL(A) is #P-hard. We believe that this is known (implicitly it is underlying [8]), but in absence of a clear reference we give a proof which boils down to a reduction from the problem of counting MAXCUTs of a graph.

Step 3(b): All 1s together. This part of the proof is the hardest, and it is difficult to describe on a high level. We assume that all entries of B are positive and that a principal submatrix in the upper left corner of B contains all 1s. We define a sequence $B^{[k]}$, for $k \geq 1$, of matrices that are obtained from B by some construction on the instances that is remotely similar to "stretching" and "thickening" (cf. [13, 5]), but more complicated. We show that WCSP($B^{[k]}$) is reducible to WCSP(B) for all k.

The entries of the $B^{[k]}$ are polynomials with integer coefficients (no longer just powers of X as the entries of B). Employing a little bit of complex analysis, we prove that for some k, $B_{11}^{[k]}$ has an irreducible factor $p(X)$ such that the multiplicity of $p(X)$ in $B_{11}^{[k]}$ is higher than in all other entries in the first row and column, and the multiplicity in the corresponding diagonal entries is also sufficiently high. Using similar tricks as in Step 1, we can filter out the powers of this irreducible polynomial $p(X)$. We obtain a matrix whose weighted CSP is #P-complete by the results of Steps 2 and 3(a).

5.1 From numbers to polynomials

We need more general versions of some results of [5]. First of all we show that a transformation similar to 'thickening' exists in the case of weighted CSP.

Let $B^{(\ell)}(X)$ denote the matrix whose entries are $(B_{ij}(X))^{\ell}$.

Lemma 17 *For any $\ell \geq 1$,* WCSP($B^{(\ell)}(X)$) *is polynomial time reducible to* WCSP($B(X)$).

Proof: Note that the canonical weighted template for $B^{(\ell)}(X)$ has the same domain D and the same equivalence relations α, β as the canonical weighted template for $B(X)$ and the weight function $\omega^{(\ell)}$ defined by $\omega^{(\ell)}(d) = \omega(d)^{\ell}$.

Take an instance $\mathcal{P} = (V; D; \mathcal{C})$ of WCSP($B^{(\ell)}$). Then

- replace every $v \in V$ with $v_1, \ldots, v_l$ and denote the resulting set by V';
- replace every constraint $\langle (u, v), \alpha \rangle$ with $\langle (u_1, v_1), \alpha \rangle$;
- replace every constraint $\langle (u, v), \beta \rangle$ with $\langle (u_1, v_1), \beta \rangle$;
- for $v \in V$ and $1 \leq i, j \leq l$, include the constraints $\langle (v_i, v_j); \alpha \rangle$, $\langle (v_i, v_j); \beta \rangle$;
- denote the resulting set of constraints by $\mathcal{C}'$ and the problem $(V'; D; \mathcal{C}')$ by $\mathcal{P}'$.

Clearly, $\mathcal{P}'$ is an instance of WCSP(B) with $\mathcal{Z}_B(\mathcal{P}) = \mathcal{Z}_{B^{(\ell)}}(\mathcal{P}')$. ■

A proof of the following lemma is straightforward.

Lemma 18 *Let $B(X) \in \mathbb{Q}[X]^{m\times n}$ be a matrix and a an integer. Then* WCSP$(B(a))$*, where $B(a)$ denotes the numerical matrix obtained from $B(X)$ by substituting a for X, is polynomial time reducible to* WCSP$(B(X))$*. Moreover, a can be chosen such that* $\mathsf{rank}(B(a)) = \mathsf{rank}(B(X))$*.*

For a matrix $B \in \mathbb{S}^{m\times n}$ and an instance $\mathcal{P} = (V; D; \mathcal{C})$ of WCSP(B), we let

$$P_B(\mathcal{P}) = \Big\{ \prod_{\substack{1\le i\le m\\ 1\le j\le n}} (B)_{ij}^{m_{ij}} \mid 0 \le m_{ij} \le |V| \text{ for } 1 \le i \le m, 1 \le j \le n \text{ such that } \textstyle\sum_{\substack{1\le i\le m\\ 1\le j\le n}} m_{ij} = |V| \Big\}.$$

Then

$$\{\omega_B(\sigma) \mid \sigma \text{ is a solution of } \mathcal{P}\} \subseteq P_B(\mathcal{P}).$$

Note that for fixed B the size of $P_B(\mathcal{P})$ is polynomial in the size of $\mathcal{P}$ and that $P_B(\mathcal{P})$ can be computed in polynomial time. Also note that

$$\mathcal{Z}_B(\mathcal{P}) = \sum_{w\in P_B(\mathcal{P})} w \cdot N_B(\mathcal{P}, w), \tag{3}$$

where $N_B(\mathcal{P}, w)$ denotes the number of solutions σ of $\mathcal{P}$ such that $\omega_B(\sigma) = w$. Let COUNT(B) denote the following problem:

Input: WCSP(B)-instance $\mathcal{P}$, $w \in \mathbb{S}$.
Objective: Compute $N_B(\mathcal{P}, w)$.

Lemma 19 *Let $B \in \mathbb{S}^{m\times n}$. Then the problems* WCSP(B) *and* COUNT(B) *are polynomial time equivalent.*

Proof: The following proof mainly follows the proof of Lemma 3.3 from [5]. As is noticed above the set $P_B(\mathcal{P})$ can be constructed in polynomial time. Therefore, if COUNT(B) is polynomial time solvable, then, by (3), so is WCSP(B).

To prove the converse, let $\mathcal{P}, w$ be an instance of COUNT(B). Suppose that $w_1, \dots, w_t$ are the non-zero elements of $P_B(\mathcal{P})$. If $w \notin P_B(\mathcal{P})$ then $N_B(\mathcal{P}, w) = 0$. Assume now that $w = w_j \in P_B(\mathcal{P})$. For $1 \le \ell \le t$, consider the number $\mathcal{Z}_{B^{(\ell)}}(\mathcal{P})$. We have

$$\mathcal{Z}_{B^{(\ell)}}(\mathcal{P}) = \sum_{\substack{\sigma \text{ a solution}\\ \text{to } \mathcal{P}}} \prod_{v\in V} \omega_B^{\ell}(v) = \sum_{w\in P_B(\mathcal{P})} w^{\ell} N_B(\mathcal{P}, w).$$

If $\mathbb{S}$ is a numerical ring then we complete the proof as in [5]. If $\mathbb{S}$ is a polynomial ring then denote the value $\mathcal{Z}_{B^{(\ell)}}(\mathcal{P})$ by $f_\ell(X)$ and notice the equations above can be represented in the matrix form

$$\begin{pmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_t(X) \end{pmatrix} = \begin{pmatrix} w_1(X) & w_2(X) & \cdots & w_t(X) \\ w_1^2(X) & w_2^2(X) & \cdots & w_t^2(X) \\ \vdots & \vdots & & \vdots \\ w_1^t(X) & w_2^t(X) & \cdots & w_t^t(X) \end{pmatrix} \cdot \begin{pmatrix} N_B(\mathcal{P}, w_1) \\ N_B(\mathcal{P}, w_2) \\ \vdots \\ N_B(\mathcal{P}, w_t) \end{pmatrix}. \tag{4}$$

On the one hand, the determinant of the square matrix is Vandermonde. Since all $w_1, \dots, w_t$ are distinct and non-zero, it is also non-zero. On the other hand, this determinant is a non-zero polynomial; let us denote it by $d(X)$. Therefore, there is an integer

$$a \le t \cdot \max_{1 \le j \le t} \mathsf{deg}(w_j) + 1$$

such that $d(a) \neq 0$. Substituting a instead of X in (4), we obtain a numerical matrix equation of the form

$$\begin{pmatrix} f_1(a) \\ f_2(a) \\ \vdots \\ f_t(a) \end{pmatrix} = \begin{pmatrix} w_1(a) & w_2(a) & \cdots & w_t(a) \\ w_1^2(a) & w_2^2(a) & \cdots & w_t^2(a) \\ \vdots & \vdots & & \vdots \\ w_1^t(a) & w_2^t(a) & \cdots & w_t^t(a) \end{pmatrix} \cdot \begin{pmatrix} N_B(\mathcal{P}, w_1) \\ N_B(\mathcal{P}, w_2) \\ \vdots \\ N_B(\mathcal{P}, w_t) \end{pmatrix}.$$

with a regular matrix, which we can solve to find the desired value $N_B(\mathcal{P}, w_j)$. ∎

Lemma 20 *(1) Let $B \in \mathbb{Q}^{k\times\ell}$ be a non-negative matrix with at least one block of rank at least 2. Then there is a non-negative matrix $C \in \mathbb{Z}^{k\times\ell}$ satisfying the same condition and such that* WCSP(C) *is polynomial time reducible to* WCSP(B).

(2) Let $B(X) \in \mathbb{Q}[X]^{k\times\ell}$ be a non-negative matrix with at least one block of rank at least 2. Then there is a non-negative matrix $C(X) \in \mathbb{Z}[X]^{k\times\ell}$ satisfying the same condition and such that WCSP$(C(X))$ *is polynomial time reducible to* WCSP$(B(X))$.

Proof: Let N be the least common denominator of entries of B [of coefficients of entries of $B(X)$]. We set C to be the matrix with entries $C_{ij} = N \cdot B_{ij}$. Since for any WCSP(B)-instance $\mathcal{P}$

$$\mathcal{Z}_C(\mathcal{P}) = N^{|V|} \mathcal{Z}_B(\mathcal{P}),$$

the problems WCSP(B) and WCSP(C) are polynomial time equivalent. ∎

Lemma 21 *(1) Let $B \in \mathbb{Z}^{m\times n}$ be a non-negative matrix, and p a prime number. Let C be the matrix obtained from B by replacing all entries divisible by p with* 0. *Then there is a polynomial time reduction from* WCSP(C) *to* WCSP(B).

(2) Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a matrix, and $p(X)$ an irreducible polynomial. Let $C(X)$ be the matrix obtained from $B(X)$ by replacing all entries divisible by p with 0. *Then there is a polynomial time reduction from* WCSP$(C(X))$ *to* WCSP$(B(X))$.

Proof: We first prove (2) by reducing WCSP($C(X)$) to COUNT($B(X)$); this is sufficient by Lemma 19. Given an instance $\mathcal{P} = (V; D; \mathcal{C})$ of WCSP($C(X)$), we first compute the set $P_C(\mathcal{P}) = (P_B(\mathcal{P}) - \{w \mid w \text{ divisible by } p\}) \cup \{0\}$. Then for each $w \in P_C(\mathcal{P}) - \{0\}$, we compute the number $N_B(\mathcal{P}, w)$ using an oracle to COUNT(B). Then we compute

$$\mathcal{Z}_C(\mathcal{P}) = \sum_{w \in P_C(\mathcal{P}) - \{0\}} N_B(\mathcal{P}, w) \cdot w.$$

For (1) the proof is quite similar. ∎

Let p be a prime number [an irreducible polynomial]. For an integer $a \in \mathbb{Z}$ [a polynomial $a \in \mathbb{Z}[X]$] we let

$$a|_p = \begin{cases} p^{\max\{k \mid k \geq 0,\ p^k \text{ divides } a\}}, & \text{if } a \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For a matrix $B \in \mathbb{Z}^{m \times n}$ [a matrix $B \in \mathbb{Z}[X]^{m \times n}$] we let $B|_p$ be the matrix with entries $(B|_p)_{ij} = (B_{ij})|_p$.

Lemma 22 *(1) Let $B \in \mathbb{Z}^{m \times n}$ be a non-negative matrix, and p a prime number. Then* WCSP($B|_p$) *is polynomial time reducible to* WCSP(B).

(2) Let $B(X) \in \mathbb{Z}[X]^{m \times n}$ be a matrix, and $p(X)$ an irreducible polynomial. Then WCSP($B(X)|_p$) *is polynomial time reducible to* WCSP($B(X)$).

Proof: We prove part (2) of the lemma. For (1) the proof is similar. Let ω be the weight function corresponding B and ω' the one corresponding to $B|_p$. Let $\mathcal{P}$ be an instance of WCSP($B(X)|_p$). Note that for every solution σ of $\mathcal{P}$ we have $\omega'(\sigma) = (\omega(\sigma))|_p$. Then by (3),

$$\mathcal{Z}_{B|_p}(\mathcal{P}) = \sum_{w \in P_B(\mathcal{P})} w|_p \cdot N_B(\mathcal{P}, w).$$

Thus WCSP($B|_p$) is polynomial time reducible to COUNT(B) and hence, by Lemma 19, to WCSP(B). ∎

Lemma 23 *(1) Let $B \in \mathbb{Z}^{m \times n}$ be a non-negative matrix which has a block of rank at least 2. Then there is some prime number p such that there is a block of $B|_p$ of rank at least 2.*

(2) Let $B(X) \in \mathbb{Z}[X]^{m \times n}$ be a non-negative matrix which has a block of rank at least 2. Then there is some irreducible polynomial $p(X)$ such that there is a block of $B|_p$ of rank at least 2.

Proof: As usual, we prove only (2), because for (1) the proof is similar. Suppose that for all irreducible p every block of the matrix $B|_p$ has rank at most 1. We shall prove that any two rows from the same block of B are linearly dependent, which is impossible.

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be two rows from a block of rank 2 of B. Then we have

$$a_i = \prod_p a_i|_p \quad \text{and} \quad b_i = \prod_p b_i|_p$$

for all i, where the (finite) products are taken over all irreducible p dividing an entry of B. Since for every irreducible p the rank of $B|_p$ is 1, there is λ_p, μ_p such that $\lambda_p a_i|_p = \mu_p b_i|_p$ for $1 \leq i \leq n$. Let

$$\Lambda = \prod_p \lambda_p \quad \text{and} \quad M = \prod_p \mu_p.$$

Then

$$\Lambda a_i = \Lambda \cdot \prod_p a_i|_p = \prod_p \lambda_p a_i|_p = \prod_p \mu_p b_i|_p = M \cdot \prod_p b_i|_p = M b_i$$

for $1 \leq i \leq n$. This shows that indeed $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent. ■

Lemma 24 *Let $p(X)$ be an irreducible polynomial, $B \in \mathbb{Z}[X]^{m \times n}$ a non-negative matrix whose entries are either 0 or powers of $p(X)$ and $a \in \mathbb{Z}$ such that $B(a)$ is non-negative and $\mathsf{rank}(B(a)) = \mathsf{rank}(B(X))$. Then the problem* WCSP$(B(X))$ *is polynomial time reducible to* WCSP$(B(a))$.

Proof: Let us denote $Y = p(X)$ and $b = p(a)$; let $B'(Y)$ be such that $B'(Y) = B(X)$. Then $B'(Y)$ is a matrix whose entries are either 0 or powers of Y. Let $\ell_{\max}$ be maximum such that $Y^{\ell_{\max}}$ is an entry of $B'(Y)$. For every instance $\mathcal{P}$ of WCSP$(B(X))$ with, say, m variables, $Z_{B'(Y)}(\mathcal{P})$ is a polynomial in Y of degree at most $m \cdot \ell_{\max}$. By Lemma 17, we can evaluate this polynomial for $Y = b^i$ for all $i \geq 0$. Thus we can interpolate and compute its coefficients. ■

Corollary 25 *Let a be an integer, and B be an integer $m \times n$-matrix whose entries are either 0 or powers of a. Let $B(X)$ be the matrix obtained from B by replacing powers of a with corresponding powers of indeterminate X, that is, by letting*

$$(B(X))_{ij} = \begin{cases} X^l, & \text{if } B_{ij} = a^l \text{ for some } l \geq 0, \\ 0, & \text{if } B_{ij} = 0. \end{cases}$$

Then the problem WCSP$(B(X))$ *is polynomial time reducible to* WCSP(B).

The following corollary summarizes the results of this subsection:

Corollary 26 *Let $\mathbb{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}[X], \mathbb{Q}[X]\}$, and let $B \in \mathbb{S}^{k \times \ell}$ be a non-negative matrix that has a block of row rank at least 2.*

Then there exists a matrix $B^* \in \mathbb{Z}[X]^{k \times \ell}$ whose non-zero entries are powers of X such that B^* has a block of row rank at least 2, and WCSP(B^*) *is reducible to* WCSP(B).

We now prove the analogue of Corollary 26 for real matrices. The proof is very similar to the proof for the integers, the only difference being that we have to replace the primes involved there by suitable real numbers forming an l-basis for B.

Let $B \in \mathbb{R}^{k\times\ell}$ be a non-negative matrix. Let $a_1, \dots, a_m$ be the positive entries of B. For $1 \le i \le m$, let $r_i = \ln a_i$. We view $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and are interested in the subspace L_B generated by $r_1, \dots, r_m$. Let $1 \le i_1 < \dots < i_n \le m$ such that $r_{i_1}, \dots, r_{i_n}$ form a basis of L_B. For $1 \le j \le n$, let $b_j = a_{i_j}$. We call $\{b_1, \dots, b_n\}$ an *l-basis* for B. Note that every positive entry of B has a unique representation $b_1^{q_1} \dots b_n^{q_n}$, where $q_1, \dots, q_n \in \mathbb{Q}$. Since for every positive $s \in \mathbb{Z}$ the problem $\mathrm{WCSP}(B^{(s)})$ is reducible to $\mathrm{WCSP}(B)$, we may actually assume without loss of generality that every positive entry of B has a unique representation $b_1^{\ell_1} \dots b_n^{\ell_n}$, where $\ell_1, \dots, \ell_n \in \mathbb{Z}$. We can make this last assumption because B is fixed and not part of the input. Observe that for every instance $\mathcal{P}$ and $w \in P_B(\mathcal{P})$, $w \neq 0$, we have $\ln w \in L_B$. Thus w has a unique representation $b_1^{m_1} \dots b_n^{m_n}$ with $m_1, \dots, m_n \in \mathbb{Z}$.

Lemma 27 *Let $B \in \mathbb{R}^{k\times\ell}$ be a non-negative matrix that has a block of row rank at least 2.*

Then there exists a matrix $B^* \in \mathbb{Z}[X]^{k\times\ell}$ whose non-zero entries are powers of X such that B^* has a block of row rank at least 2, and $\mathrm{WCSP}(B^*)$ is reducible to $\mathrm{WCSP}(B)$.

Proof: Let $b^{\ell_1}, \dots, b^{\ell_n}$ be an l-basis for the entries of B. For $1 \le s \le n$ and $a = b_1^{\ell_1} \dots b_n^{\ell_n}$ we let $a|_{b_s} = b_s^{\ell_s}$, and we let $A|_{b_s}$ be the matrix with $(A|_{b_s})_{ij} = (A_{ij})|_{b_s}$ if $A_{ij} > 0$ and $(A|_{b_s})_{ij} = 0$ if $A_{ij} = 0$.

Analogously to Lemma 22 we can prove that for $1 \le s \le n$ the problem $\mathrm{WCSP}(B|_{b_s})$ is polynomial time reducible to $\mathrm{WCSP}(B)$. Here we use the fact that for every instance $\mathcal{P}$ every element of the set $P_B(\mathcal{P})$ has a unique representation in terms of our l-basis.

Analogously to Lemma 23 we can prove that if B has a block of row rank at least 2 then for some $b \in \{b_1, \dots, b_n\}$ the matrix $B|_b$ has a block of row rank at least 2.

To complete the proof of Lemma 27, assume that B has a block of row rank at least 2 and let $b \in \{b_1, \dots, b_n\}$ such that $B|_b$ also satisfies this condition. Let X be an indeterminate and B^* the matrix obtained from $B|_b$ by replacing each entry b^ℓ by X^ℓ. Let $\mathcal{P}$ be an instance with m variables. We want to compute the polynomial $q(X) = \mathcal{Z}_{B^*}(\mathcal{P})$, which is a polynomial of degree at most $m \cdot \ell$ in X, where $\ell = \max\{\ell_i \mid a_j = b_1^{\ell_1} \dots b_n^{\ell_n},\ 1 \le j \le m\}$. Observing that for $0 \le r \le m$ we have

$$\mathcal{Z}_{(B|_b)^{(r)}}(\mathcal{P}) = q(b^r),$$

we can compute the coefficients of q by Lemma 17 and interpolation. This completes the proof of Lemma 27. ∎

5.2 Basic reductions

Recall that equivalence relations α, β are said to be *permutable* if $\alpha \circ \beta = \beta \circ \alpha$.

Lemma 28 ([1]) *If α, β are equivalence relations that are not permutable, then the problem $\#\mathrm{CSP}(\{\alpha, \beta\})$ is #P-complete.*

As is easily seen, α, β are not permutable if and only if there are $1 \leq i, j \leq m$, $1 \leq k, l \leq n$ such that $B(\alpha,\beta)_{ik}, B(\alpha,\beta)_{il}, B(\alpha,\beta)_{jk} \neq 0$, but $B(\alpha,\beta)_{jl} = 0$. This property is also sufficient for #P-completeness of the weighted problem. In the following, be a

Lemma 29 *Let $B \in \mathbb{Z}[X]^{m\times n}$ be a nonnegative matrix and $1 \leq i, k \leq n$, $1 \leq j, \ell \leq n$ such that $B_{ik}(X)$, $B_{il}(X)$, $B_{jk}(X) \neq 0$ while $B_{jl}(X) = 0$. Then* $\mathrm{WCSP}(B(X))$ *is #P-complete.*

Proof: Take an integer a such that $B(a)$ is non-negative, $B_{ik}(a), B_{il}(a), B_{jk}(a) \neq 0$, and let $B' = B(a)$. Let $(D_{B'}, \{\alpha,\beta\})$ be the canonical template for B'. By Lemmas 18 and 8, the problems $\mathrm{WCSP}(B')$ and $\#\mathrm{CSP}(\{\alpha_{B'}, \beta_{B'}\})$ are polynomial time reducible to $\mathrm{WCSP}(B(X))$. Since $B'_{ik}, B'_{il}, B'_{jk} \neq 0$ and $B'_{jl} = 0$, by Lemma 28, $\#\mathrm{CSP}(\{\alpha_{B'}, \beta_{B'}\})$ is #P-complete. ∎

In the following, we will expand our constraint language of two equivalence relations by additional unary relations. Let $(D, \{\alpha,\beta\}, \omega)$ be the canonical weighted template for a matrix $B \in \mathbb{Z}[X]^{m\times n}$. For $d \in D$, we use κ_d to denote the unary one-element relation $\{(d)\}$ and $\mathrm{CON}(D)$ to denote the set $\{\kappa_d \mid d \in D\}$. If $m = n$ then we also use the relation $\theta = \{(i,i) \mid 1 \leq i \leq m\} \subseteq D = \{1,\dots,m\} \times \{1,\dots,n\}$ consisting of all 'diagonal' elements of the canonical template. We shall use $\mathrm{WCSP}^+(B)$ to denote the problem $\mathrm{WCSP}(\{\alpha,\beta\} \cup \mathrm{CON}(D) \cup \{\theta\}, \omega)$ and $\mathrm{WCSP}^*(B)$ to denote the problem $\mathrm{WCSP}(\{\alpha,\beta\} \cup \mathrm{CON}(D), \omega)$.

As is easily seen, for an instance $\mathcal{P} = (V; D; \mathcal{C})$ of the problem $\mathrm{WCSP}^+(B)$ or the problem $\#\mathrm{CSP}(\{\alpha,\beta\} \cup \mathrm{CON}(D) \cup \{\theta\})$, specifying a set of constraints of the form $\langle (v), \kappa_d \rangle$ is equivalent to specifying a partial solution $\varphi_{\mathcal{P}}$ of $\mathcal{P}$, where $\varphi_{\mathcal{P}}(v)$ is defined to be d on every variable occurring in a constraint $\langle (v), \kappa_d \rangle$. Analogously, the set of constraints of the form $\langle (v), \theta \rangle$ can be represented by just a subset $V_{\mathcal{P}} \subseteq V$ of all variables to which such a constraint applies. To simplify the notation we will use $\mathcal{Z}_B^{\varphi_{\mathcal{P}}, V_{\mathcal{P}}}(\mathcal{P})$ to denote the objective value for an instance of $\mathrm{WCSP}^+(B)$, or $\mathcal{Z}_B^{\varphi_{\mathcal{P}}}(\mathcal{P})$ or $\mathcal{Z}_B^{V_{\mathcal{P}}}(\mathcal{P})$ if $\mathcal{P}$ is an instance of $\mathrm{WCSP}^*(B)$ or $\mathrm{WCSP}(\{\alpha,\beta\} \cup \{\theta\})$ respectively. First, we need several lemmas similar to those from the previous section.

Lemma 30 *Let $B \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix. For any $\ell \geq 1$,* $\mathrm{WCSP}^+(B^{(\ell)})$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B(X))$.

Proof: To obtain the required reduction we use the reduction from Lemma 17 and extend it by replacing unary constraints $\langle (v), \kappa_d \rangle$ and $\langle (v), \theta \rangle$ with $\langle (v_1), \kappa_d \rangle$ and $\langle (v_1), \theta \rangle$. ∎

Lemma 31 *Let $B(X) \in \mathbb{Q}[X]^{m\times n}$ be a matrix and a an integer. Then* $\mathrm{WCSP}^+(B(a))$*, where $B(a)$ denotes the numerical matrix obtained from $B(X)$ by substituting a for X, is polynomial time reducible to* $\mathrm{WCSP}^+(B(X))$*, and* $\mathrm{WCSP}^*(B(a))$ *is polynomial time reducible to* $\mathrm{WCSP}^*(B(X))$*. Moreover, a can be chosen such that* $\mathsf{rank}(B(a)) = \mathsf{rank}(B(X))$.

Proof: Analogously to the proof of Lemma 18.

Lemma 32 *Let $p(X)$ be an irreducible polynomial, $B \in \mathbb{Z}[X]^{m\times n}$ a non-negative matrix whose entries are either 0 or powers of $p(X)$ and $a \in \mathbb{Z}$ such that $B(a)$ is non-negative and $\mathsf{rank}(B(a)) = \mathsf{rank}(B(X))$. Then the problem $\mathrm{WCSP}^{+}(B(X))$ is polynomial time reducible to $\mathrm{WCSP}^{+}(B(a))$ and $\mathrm{WCSP}^{*}(B(X))$ is polynomial time reducible to $\mathrm{WCSP}^{*}(B(a))$.*

Proof: Analogously to the proof of Lemma 24.

Lemma 33 *Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix. Then $\mathrm{WCSP}^{*}(B)$ is polynomial time reducible to $\mathrm{WCSP}(B)$.*

Proof: By Lemmas 18, 24 and 31, 32, there is a non-negative matrix $C \in \mathbb{Z}^{m\times n}$ such that $\mathrm{WCSP}(B(X))$ and $\mathrm{WCSP}(C)$, and $\mathrm{WCSP}^{*}(B(X))$ and $\mathrm{WCSP}^{*}(C)$ are polynomial time equivalent. Let $(D_C, \{\alpha_C, \beta_C\})$ denote the canonical template for C. By Corollary 8, $\mathrm{WCSP}(C)$ is equivalent to $\#\mathrm{CSP}(\{\alpha_C, \beta_C\})$. By Theorem 12 of [1], the problem $\#\mathrm{CSP}(\{\alpha_C, \beta_C\} \cup \mathrm{CON}(D_C))$ is polynomial time reducible to $\#\mathrm{CSP}(\{\alpha_C, \beta_C\})$.

We show that $\mathrm{WCSP}^{*}(C)$ is polynomial time reducible to $\#\mathrm{CSP}(\{\alpha_B, \beta_B\} \cup \mathrm{CON}(D_C))$. Let $\mathcal{P} = (V; D_C; \mathcal{C})$ be an instance of $\mathrm{WCSP}^{*}(C)$. For every element $d \in D_C$, say, from α-class C_i and β-class D_j, we denote the $\alpha_C \cap \beta_C$-class $C_i \cap D_j$ by $d^{\alpha_C \cap \beta_C}$. If κ_d occurs amongst the constraint relations of $\mathcal{P}$, then we may assume that this class is non-empty, because otherwise $Z_C^{\varphi_{\mathcal{P}}}(\mathcal{P}) = 0$. Fix an element d^* from this class. Then we define an instance $\mathcal{P}' = (V \cup V'; D_C; \mathcal{C}')$ of $\#\mathrm{CSP}(\{\alpha_C, \alpha_C\} \cup \mathrm{CON}(D_C))$ as follows.

- $V' = \{v' \mid v \in V_{\mathcal{P}}\}$;
- for every constraint of the form $\langle (u_1, u_2), \alpha\rangle, \langle (v_1, v_2), \beta\rangle \in \mathcal{C}$, include the constraints $\langle (u_1, u_2), \alpha_C\rangle$, $\langle (v_1, v_2), \beta_C\rangle$ into $\mathcal{C}'$;
- for every $\langle (v), \kappa_d\rangle \in \mathcal{C}$, include $\langle (v, v'), \alpha_C\rangle$, $\langle (v, v'), \beta_C\rangle$ and $\langle (v'), \kappa_{d^*}\rangle$ into $\mathcal{C}'$.

As is easily seen, for any solution $\sigma : V \to D_C$ to $\mathcal{P}'$, the following conditions hold:

- $\sigma(v') = (\varphi(v))^*$ for every $v \in V$ restricted with a constraint $\langle (v), \kappa_d\rangle$;
- $(\sigma(v), (\varphi(v))^*) \in \alpha_B \cap \beta_B$ for every $v \in V$ restricted with a constraint $\langle (v), \kappa_d\rangle$;
- the restriction of σ onto V is a solution to $\mathcal{P}$.

The last two conditions imply that $Z_C^{\varphi_{\mathcal{P}}}(\mathcal{P})$ equals the number of solutions to $\mathcal{P}'$. ∎

Lemma 34 *Let $B \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix and $C = B \cdot B^{\top}$. Then*

(1) C is a symmetric non-negative matrix.

(2) $\mathsf{rank}(C) = \mathsf{rank}(B)$.

(3) If B has a block of rank at least 2 then C also has such a block.

(4) $\mathrm{WCSP}^+(C)$ *is polynomial time reducible to* $\mathrm{WCSP}^*(B)$.

Proof: Since C is a Gram matrix of the collection of rows of B, its rank equals $\mathsf{rank}(B)$. Notice that the same holds for every block of C. Therefore, for any block of B, the matrix C has a block of the same rank. This proves (1)–(3).

To prove (4), let $\mathcal{D}_B = (D, \alpha, \beta, \omega)$ and $\mathcal{D}_C = (D'; \alpha', \beta', \omega')$ be the canonical weighted templates for B and C, respectively. Recall that $D = \{1, \dots, m\} \times \{1, \dots, n\}$ and $D' = \{1, \dots, m\}^2$. Also recall that the rows of B $[C]$ correspond to the equivalence classes of α $[\alpha'$, respectively$]$ and the columns of B $[C]$ correspond to the equivalence classes of β $[\beta'$, respectively$]$.

Take an instance $\mathcal{P}' = (V'; D'; \mathcal{C}')$ of $\mathrm{WCSP}^+(C)$. We define an instance $\mathcal{P} = (V; D; \mathcal{C})$ of $\mathrm{WCSP}^*(B)$ ($=\mathrm{WCSP}(\{\alpha, \beta\} \cup \mathrm{CON}(D))$) as follows:

(i) $V = \{v^1, v^2 \mid v \in V'\} \cup \{v^3, v^4 \mid \langle v, \kappa_{(i,j)} \rangle \in \mathcal{C}'\}$;

(ii) for every constraint $\langle (v, w), \alpha' \rangle \in \mathcal{C}'$, we include $\langle (v^1, w^1), \alpha \rangle$ into $\mathcal{C}$, and for every constraint $\langle (v, w), \beta' \rangle \in \mathcal{C}'$, we include $\langle (v^2, w^2), \alpha \rangle$ into $\mathcal{C}$;

(iii) for every $v \in V'$ we include $\langle (v^1, v^2), \beta \rangle$ into $\mathcal{C}$;

(iv) for every constraint $\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'$, we include the constraints $\langle (v^3), \kappa_{(i,1)} \} \rangle$, $\langle (v^4), \kappa_{(j,1)} \} \rangle$, $\langle (v^1, v^3), \alpha \rangle$, $\langle (v^2, v^4), \alpha \rangle$ into $\mathcal{C}$;

(v) for every constraint $\langle (v), \theta \rangle \in \mathcal{C}'$, we include $\langle (v^1, v^2), \alpha \rangle$ into $\mathcal{C}$.

To understand this, observe that the constraints in (ii) say that whenever $v, w \in V'$ are forced to be mapped to the same row of C, then v^1 and w^1 are forced to the same row of B. If v and w are forced to the same column of C, then v^2 and w^2 are forced to the same row(!) of B. The constraints in (iii) force v^1 and v^2 to the same column of B. The constraints in (iv) say that if $v \in V'$ is forced to the (i, j)-entry of C, then v^3 is forced to the first element of the ith row and v^4 is forced to the first element of the jth row. Finally, (v) says that if $v' \in V$ is forced onto the diagonal, then v^1 and v^2 are forced to the same row; since by (iii) they are also forced to the same column, this implies that they are forced to the same position.

Now let us try to understand how solutions for $\mathcal{P}'$ relate to solutions for $\mathcal{P}$. Observe first that the domain of $\varphi_{\mathcal{P}}$, that is, the variables that are fixed by constraints κ_d, is the set of all variables of the form v^3, v^4. Note that variables v^3, v^4 are only added for those $v \in V'$ that occur in some constraint $\langle v, \kappa_{(i,j)} \rangle$.

Let σ' be a solution of $\mathcal{P}'$. Let $\Psi(\sigma')$ be the set of all solutions σ for $\mathcal{P}$ such that for all $v \in V'$, if $\sigma'(v) = (i, j)$, then $\sigma(v^1) = (i, k)$ and $\sigma(v^2) = (j, k)$ for some $k \in \{1, \dots, n\}$. Then the sets $\Psi(\sigma')$, where σ' ranges over all solutions of $\mathcal{P}'$, form a partition of the space of solutions of $\mathcal{P}$. To see that every solution σ of $\mathcal{P}$ belongs to some $\Psi(\sigma')$, just recall that by (iii), v^1 and v^2 must be mapped to the same column.

More formally, for every solution σ of $\mathcal{P}$ we define a mapping $\sigma' : V' \to D'$ by letting, for every $v \in V'$,

$$\sigma'(v) = (i, j) \text{ if } \sigma(v^1) = (i, k) \text{ and } \sigma(v^2) = (j, k).$$

This mapping is well-defined on V', because by the constraints (iii) there always exists a suitable k. Then, clearly, $\sigma \in \Psi(\sigma')$. To see that σ' is a solution of $\mathcal{P}'$, note the following:

- For every constraint $\langle (u,v), \alpha' \rangle \in \mathcal{C}'$, the constraint $\langle (u^1, v^1), \alpha \rangle$ implies that $\sigma(u^1), \sigma(v^1)$ are in the same row, say, i. Hence, $(\sigma'(u), \sigma'(v)) = ((i,j),(i,k)) \in \alpha'$ for certain $1 \le j,k \le m$.
- For every constraint $\langle (u,v), \beta' \rangle \in \mathcal{C}'$, the constraint $\langle (u^2, v^2), \alpha \rangle$ implies that $\sigma(u^2), \sigma(v^2)$ are in the same row, say, j. Therefore, $(\sigma'(u), \sigma'(v)) = ((i,j),(k,j)) \in \beta'$ for certain $1 \le i,k \le m$.
- For every constraint $\langle v, \theta \rangle \in \mathcal{C}'$, the constraint $\langle (v^1, v^2), \alpha \rangle$ implies that $\sigma(v^1) = \sigma(v^2)$ are in the same row, say, i. Hence $\sigma'(v) = (i,i) \in \theta$.
- For every v such that $\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'$, the constraints $\langle (v^3), \kappa_{(i,1)}\} \rangle$, $\langle (v^4), \kappa_{(j,1)}\} \rangle$ imply $\sigma(v^3) = (i,1)$ and $\sigma(v^4) = (j,1)$. Then the constraints $\langle (v^1, v^3), \alpha \rangle$, $\langle (v^2, v^4), \alpha \rangle$ imply that $\sigma(v^1)$ is in row i and $\sigma(v^2)$ in row j. Thus $\sigma'(v) = (i,j) \in \kappa_{(i,j)}$.

Finally, we have

$$\begin{aligned}
\mathcal{Z}_B^{\varphi_{\mathcal{P}}}(\mathcal{P}) &= \left(\prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} B_{i1} B_{j1} \right) \cdot \left(\sum_{\substack{\sigma' \text{ solution} \\ \text{to } \mathcal{P}'}} \sum_{\sigma \in \Psi(\sigma')} \prod_{\substack{v \in V' \\ \sigma'(v) = (i,j) \\ \sigma(v^1) = (i,k), \sigma(v^2) = (j,k)}} B_{ik} B_{jk} \right) \\
&= \left(\prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} B_{i1} B_{j1} \right) \cdot \left(\sum_{\substack{\sigma' \text{ solution} \\ \text{to } \mathcal{P}'}} \prod_{\substack{v \in V' \\ \sigma'(v) = (i,j)}} \sum_{k=1}^{n} B_{ik} B_{jk} \right) \\
&= \left(\prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} B_{i1} B_{j1} \right) \cdot \left(\sum_{\substack{\sigma' \text{ solution} \\ \text{to } \mathcal{P}'}} \prod_{\substack{v \in V' \\ \sigma'(v) = (i,j)}} C_{ij} \right) \\
&= \left(\prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} B_{i1} B_{j1} \right) \cdot \mathcal{Z}_C^{\varphi_{\mathcal{P}'}, V_{\mathcal{P}'}}(\mathcal{P}').
\end{aligned}$$

Since the term $\prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} B_{i1} B_{j1}$ can easily be computed in polynomial time, this yields the desired reduction. ∎

Next we show that the reductions established in Lemmas 19 and 22 work for the more general problem WCSP$^+(B)$. By COUNT$^+(B)$ we denote the problem of finding the number $N_B^+(\mathcal{P}, w)$ of solutions σ to a WCSP$^+(B)$ instance $\mathcal{P}$ such that $\omega_B(\sigma) = w$. The following lemmas can be proved using the same arguments as in the proofs of Lemmas 19 and 22. Instead of Lemma 17, we use Lemma 30.

Lemma 35 *Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix. Then the problems* $\mathrm{WCSP}^+(B(X))$ *and* $\mathrm{COUNT}^+(B(X))$ *are polynomial time equivalent.*

Lemma 36 *(1) Let $B(X) \in \mathbb{Z}^{m\times n}$ be a non-negative matrix, and p a prime number. Then* $\mathrm{WCSP}^+(B|_p)$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B)$ *and* $\mathrm{WCSP}^*(B|_p)$ *is polynomial time reducible to* $\mathrm{WCSP}^*(B)$.

(2) Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix, and $p(X)$ an irreducible polynomial. Then $\mathrm{WCSP}^+(B(X)|_p)$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B(X))$ *and* $\mathrm{WCSP}^*(B(X)|_p)$ *is polynomial time reducible to* $\mathrm{WCSP}^*(B(X))$.

Finally, analogously to Corollary 25 we obtain:

Lemma 37 *Let $p(X) \in \mathbb{Z}[X]$ and $B \in \mathbb{Z}[X]^{m\times n}$ such that all entries of B are either 0 or powers of $p(X)$. Let $B(Y)$ be the matrix obtained from B by replacing powers of $p(X)$ with corresponding powers of indeterminate Y, that is, by letting*

$$(B(Y))_{ij} = \begin{cases} Y^l, & \text{if } B_{ij} = p(X)^l \text{ for some } l \ge 0, \\ 0, & \text{if } B_{ij} = 0. \end{cases}$$

Then the problem $\mathrm{WCSP}^+(B(Y))$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B)$.

Note that if a symmetric matrix B is not connected then by deleting rows with all entries 0 and permuting rows and columns, it can be made *quasi-diagonal*, that is, of the form displayed in Figure 1. Of course the blocks $B_1, \dots, B_k$ may be of different sizes.

$$\left(\begin{array}{c|c|c|c} B_1 & 0 & \cdots & 0 \\ \hline 0 & B_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & B_k \end{array}\right)$$

Figure 1: A quasi-diagonal matrix B

Lemma 38 *Let $B(X) \in \mathbb{Z}[X]^{m\times m}$ be a non-negative quasi-diagonal symmetric matrix with blocks $B_1, \dots, B_k$ (as in Figure 1).*

(1) For any $r \in \{1, \dots, k\}$ and any $p, q \in \{1, \dots, m\}$ such that uth row and vth column cross B_r, $\mathrm{WCSP}^+(C)$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B)$*, where C has the same size as B_r and $C_{ij} = (B_r)_{ij} \cdot (B_r)_{iq} \cdot (B_r)_{pj}$.*

(2) If B_r is positive and $\mathsf{rank}(B_r) \ge 2$ then $\mathsf{rank}(C) \ge 2$.

Proof: Without loss of generality we assume that $r = 1$. Suppose that $B_1 \in \mathbb{Z}[X]^{n\times n}$ for some $n \leq m$.

To prove (1), let $\mathcal{D} = (D;\alpha,\beta,\omega)$ and $\mathcal{D}' = (D';\alpha',\beta',\omega')$ be the canonical weighted template for C and B, respectively. Note that

$$D = \{1,\dots,n\}^2 \subseteq \{1,\dots,m\}^2 = D'.$$

Furthermore, let θ be the diagonal of D and θ' the diagonal of D'. The κ_d for $d \in D$ are the same on both D and D'.

Let $\mathcal{P} = (V;D;\mathcal{C})$ be an instance of WCSP$^+(C)$. We transform it to an instance $\mathcal{P}' = (V';D';\mathcal{C}')$ of WCSP$^+(B(X))$ as follows:

(i) $V' = V \cup \{v^1, v^2 \mid v \in V\} \cup \{x\}$.

(ii) For every constraint from $\mathcal{C}$ we include into $\mathcal{C}'$ the analogous constraint replacing α,β,θ by α',β',θ', respectively, and κ_d by κ_d.

(iii) For every $v \in V$ we include into $\mathcal{C}'$ the constraints $\langle (v,v^1),\alpha\rangle$, $\langle (v,v^2),\beta\rangle$, $\langle (v^1,x),\beta\rangle$, $\langle (v^2,x),\alpha\rangle$.

(iv) We include the constraint $\langle x, \kappa_{(p,q)}\rangle$.

Note that the constraints (iii) and (iv) guarantee that x is forced to (p,q), and for every $v \in V$, v^1 is forced into the same row as v and column q, and v^2 is forced into row p and the same column as v.

Now let σ be a solution of WCSP$^+(C)$. Define

$$\sigma' : V \cup \{v^1, v^2 \mid v \in V\} \cup \{x\} \to D$$

as follows: For every $v \in V$, let $\sigma'(v) = \sigma(v)$. If $\sigma(v) = (i,j)$, let $\sigma'(v^1) = (i,q)$ and $\sigma'(v^2) = (p,j)$. Finally, let $\sigma'(x) = (p,q)$. Then σ' is a solution of WCSP$^+(B)$.

Conversely, if σ' is a solution of WCSP$^+(B)$ of non-zero weight $\omega'(\sigma')$, then $\sigma'(x) = (p,q)$, and for all $v \in V$ with $\sigma'(v) = (i,j)$ we have $\sigma'(v^1) = (i,q)$ and $\sigma'(v^2) = (p,j)$. Now $\omega'(\sigma') \neq 0$ implies that $(i,j) \in D$, because all entries of row p and column q of B that are outside of B_1 are zero.

Thus there is a one-to-one correspondence between solutions σ of WCSP$^+(C)$ and solutions σ' of WCSP$^+(B)$ of non-zero weight.

For every $\sigma : V \to D$, let σ_1 and σ_2 the projection of σ on the first and second component, respectively, that is, if $\sigma(v) = (i,j)$, then $\sigma_1(v) = i$ and $\sigma_2(v) = j$. Then

$$\omega'(\sigma') = B_{pq} \cdot \prod_{v\in V} B_{\sigma_1(v)\sigma_2(v)} \cdot B_{\sigma_1(v)q} \cdot B_{p\sigma_2(v)} = B_{pq} \cdot \omega(\sigma).$$

Thus the mapping $\mathcal{P} \mapsto \mathcal{P}'$ yields the desired reduction from WCSP$^+(C)$ to WCSP$^+(B)$ and thus proves (1).

To prove (2), suppose that the following determinant is non-zero:

$$\begin{vmatrix} B_{ij} & B_{ij'} \\ B_{i'j} & B_{i'j'} \end{vmatrix} \neq 0.$$

Then we have

$$\left|\begin{array}{cc} C_{ij} & C_{ij'} \\ C_{i'j} & C_{i'j'} \end{array}\right| = \left|\begin{array}{cc} B_{ij}B_{iq}B_{pj} & B_{ij'}B_{iq}B_{pj'} \\ B_{i'j}B_{i'q}B_{pj} & B_{i'j'}B_{i'q}B_{pj'} \end{array}\right| = B_{iq}B_{i'q}B_{pj}B_{pj'} \cdot \left|\begin{array}{cc} B_{ij} & B_{ij'} \\ B_{i'j} & B_{i'j'} \end{array}\right| \neq 0.$$

■

Making use of Lemmas 34 and 36 we may assume that our matrix B is symmetric and every entry of B is a power of an indeterminate X. Then Lemma 29 allows us to assume that all blocks of B are positive, finally by Lemma 38 B can be assumed connected. We show that in some cases the degrees of powers of X in B can be decreased.

Lemma 39 *Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix, and every entry of $B(X)$ divisible by X. Then* WCSP$^+(\frac{1}{X}B(X))$ *is polynomial time reducible to* WCSP$^+(B(X))$.

Proof: Take $\mathcal{P} = (V; D; \mathcal{C}) \in$ WCSP$^+(\frac{1}{X}B(X))$. Then

$$\mathcal{Z}^{\varphi_{\mathcal{P}},V_{\mathcal{P}}}_{\frac{1}{X}B(X)}(\mathcal{P}) = \frac{1}{X^{|V|}}\mathcal{Z}^{\varphi_{\mathcal{P}},V_{\mathcal{P}}}_{B}(\mathcal{P}).$$

■

A row [column] of the matrix $B(X)$ is called an *1-row* [*1-column*] if 1 occurs in it.

Lemma 40 *Let $B(X) \in \mathbb{Z}[X]^{m\times m}$ be a symmetric non-negative matrix whose entries are powers of the indeterminant X, and let $C(X)$ be the minor of $B(X)$ obtained by removing all non-1-rows and all non-1-columns and $A(X)$ the minor obtained by removing all non-1-rows. Then*

(1) WCSP$^+(C(X))$ *is polynomial time reducible to* WCSP$^+(B(X))$*;*

(2) WCSP$^*(A(X))$ *is polynomial time reducible to* WCSP$^*(B(X))$.

Proof: (1) Without loss of generality we may assume that the first n rows of B are the 1-rows and thus that the first n-columns are the 1-columns. Let C' denote the $n \times n$-matrix with entries $C'_{ij} = \ell_i \cdot \ell_j \cdot C_{ij}$, where ℓ_i is the number of 1s in the ith row (column) of C. We first show that WCSP$^+(C')$ is polynomial time reducible to WCSP$^+(B)$. Let $\mathcal{D} = (D, \alpha, \beta, \omega)$ and $\mathcal{D}' = (D', \alpha', \beta', \omega')$ be the canonical weighted templates of C' and B, respectively. Then $D = \{1, \dots, n\}^2$ and $D' = \{1, \dots, m\}^2$. Let θ, θ' the diagonals of D and D'.

We use Δ to denote the maximal degree of X in $C'(X)$. Let $\mathcal{P} = (V; D; \mathcal{C})$ be an instance of WCSP$^+(C'(X))$. We define an instance $\mathcal{P}' = (V'; D'; \mathcal{C}')$ of WCSP$^+(B(X))$ as follows:

(i) Let $V' = V \cup \{v_1, \dots, v_k, v^1, \dots, v^k \mid v \in V\}$, where $k = |V| \cdot \Delta + 1$.

(ii) For every constraint $\langle (u, v), \alpha\rangle \in \mathcal{C}$, add the constraint $\langle (u, v), \alpha'\rangle$ to $\mathcal{C}'$. Similarly, for every constraint $\langle (u, v), \beta\rangle \in \mathcal{C}$, add the constraint $\langle (u, v), \beta'\rangle$ to $\mathcal{C}'$, and for every constraint $\langle (v), \theta\rangle \in \mathcal{C}$, add the constraint $\langle (v), \theta'\rangle$ to $\mathcal{C}'$.

(iii) For every constraint $\langle v, \kappa_d \rangle \in \mathcal{C}$, add the constraint $\langle v, \kappa_d \rangle$ to $\mathcal{C}'$.

(iv) For every $v \in V$ and $1 \le i \le k$, add the constraints $\langle (v, v_i), \alpha \rangle$ and $\langle (v, v^i), \beta \rangle$ to $\mathcal{C}'$.

(v) For every $v \in V$ and $1 \le i < k$, add the constraints $\langle (v_i, v_{i+1}), \alpha \rangle$, $\langle (v_i, v_{i+1}), \beta \rangle$, $\langle (v^i, v^{i+1}), \alpha \rangle$, and $\langle (v^i, v^{i+1}), \beta \rangle$ to $\mathcal{C}'$.

The constraints (i)–(iii) just make sure that the restriction of a solution of $\mathcal{P}'$ to V is a solution of $\mathcal{P}$. By (iv), for every $v \in V$, the variables $v_1, \dots, v_k$ are forced to the same row as v and the variables $v^1, \dots, v^k$ to the same column. By the constraints in (v), for every $v \in V$ the variables $v_1, \dots, v_k$ are forced to the same entry, and so are the vertices $v^1, \dots, v^k$.

Let $\sigma' : V' \to D'$ be a solution of $\mathcal{P}'$. Then

$$\omega'(\sigma') = \prod_{v \in V} B_{\sigma'(v)} \cdot B^k_{\sigma'(v_1)} \cdot B^k_{\sigma'(v^1)}.$$

Observe that $\mathsf{deg}(\omega'(\sigma')) < k$ if and only if for every $v \in V$,

$$B_{\sigma'(v_1)} = B_{\sigma'(v^1)} = 1.$$

This is only possible if $\sigma'(v)$ is contained in a 1-row and in a 1-column, that is, in D. In this case, the restriction σ of σ' to V is a solution of $\mathcal{P}$. Conversely, for every solution σ of $\mathcal{P}$ there is an extension σ' that is a solution of $\mathcal{P}'$ with $\mathsf{deg}(\omega'(\sigma')) < k$. As can be easily seen, there are

$$\ell_1^{|V_1|} \cdot \ldots \cdot \ell_n^{|V_n|} \cdot \ell_1^{|V^1|} \cdot \ldots \cdot \ell_n^{|V^n|}$$

such extensions where V_i denotes the set of variables v such that $\sigma(v) = (i, j)$ for a certain j, and V^i denotes the set of variables v such that $\sigma(v) = (j, i)$ for a certain j. Therefore,

$$\sum_{\sigma \text{is the restriction of} \sigma'} \omega'(\sigma') = \omega(\sigma).$$

Thus

$$\mathcal{Z}^{\phi_\mathcal{P}, V_\mathcal{P}}_{C'(X)} = \sum_{\sigma \text{ solution of } \mathcal{P}} \omega(\sigma) = \sum_{\substack{\sigma' \text{ solution of } \mathcal{P}' \\ \mathsf{deg}(\omega'(\sigma'))<k}} \omega(\sigma').$$

This yields a reduction from WCSP$^+(C'(X))$ to COUNT$^+(B(X))$ and thus to WCSP$^+(B(X))$ by Lemma 35.

Finally, let p be a prime number that does not divide the ℓ_i. Then the entries of $C'(p)|_p$ are powers of p and $(C'(p)|_p)_{ij} = p^s$ if and only if $C(X)_{ij} = X^s$. By Lemmas 31 and 36, WCSP$^+(C'(p)|_p)$ is polynomial time reducible to WCSP$^+(C'(X))$; and, by Lemma 32, WCSP$^+(C(X))$ is polynomial time reducible to WCSP$^+(C'(p)|_p)$.

(2) The proof in this case is similar. Let A' denote the $n \times m$-matrix with entries $A'_{ij} = \ell_i \cdot A_{ij}$, where ℓ_i is the number of 1s in the ith row of A. We first show that

WCSP$^*(A')$ is polynomial time reducible to WCSP$^*(B)$. Let $\mathcal{D}'' = (D'', \alpha, \beta, \omega)$ be the canonical weighted templates of A'. Then $D'' = \{1, \dots, m\} \times \{1, \dots, n\}$.

We use Δ to denote the maximal degree of X in $A'(X)$. Let $\mathcal{P} = (V; D''; \mathcal{C})$ be an instance of WCSP$^*(A'(X))$. We define an instance $\mathcal{P}' = (V'; D'; \mathcal{C}')$ of WCSP$^*(B(X))$ as follows:

(i) Let $V' = V \cup \{v_1, \dots, v_k \mid v \in V\}$, where $k = |V| \cdot \Delta + 1$.

(ii) For every constraint $\langle (u,v), \alpha \rangle \in \mathcal{C}$, add the constraint $\langle (u,v), \alpha' \rangle$ to $\mathcal{C}'$. Similarly, for every constraint $\langle (u,v), \beta \rangle \in \mathcal{C}$, add the constraint $\langle (u,v), \beta' \rangle$ to $\mathcal{C}'$.

(iii) For every constraint $\langle v, \kappa_d \rangle \in \mathcal{C}$, add the constraint $\langle v, \kappa_d \rangle$ to $\mathcal{C}'$.

(iv) For every $v \in V$ and $1 \leq i \leq k$, add the constraints $\langle (v, v_i), \alpha \rangle$ to $\mathcal{C}'$.

(v) For every $v \in V$ and $1 \leq i < k$, add the constraints $\langle (v_i, v_{i+1}), \alpha \rangle$, $\langle (v_i, v_{i+1}), \beta \rangle$, to $\mathcal{C}'$.

As well as in part (1), the constraints (i)–(iii) just make sure that the restriction of a solution of $\mathcal{P}'$ to V is a solution of $\mathcal{P}$. By (iv), for every $v \in V$, the variables $v_1, \dots, v_k$ are forced to the same row as v. By the constraints in (v), for every $v \in V$ the variables $v_1, \dots, v_k$ are forced to the same entry.

Let $\sigma' : V' \to D'$ be a solution of $\mathcal{P}'$. Then

$$\omega'(\sigma') = \prod_{v \in V} B_{\sigma'(v)} \cdot B^k_{\sigma'(v_1)}.$$

Observe that $\mathsf{deg}(\omega'(\sigma')) < k$ if and only if for every $v \in V$,

$$B_{\sigma'(v_1)} = 1.$$

This is only possible if $\sigma'(v)$ is contained in a 1-row and in a 1-column, that is, in D''. In this case, the restriction σ of σ' to V is a solution of $\mathcal{P}$. Conversely, for every solution σ of $\mathcal{P}$ there is an extension σ' that is a solution of $\mathcal{P}'$ with $\mathsf{deg}(\omega'(\sigma')) < k$. As can be easily seen, there are

$$\ell_1^{|V_1|} \cdot \dots \cdot \ell_n^{|V_n|}$$

such extensions, where V_i denotes the set of variables v such that $\sigma(v) = (i,j)$ for a certain j. Therefore,

$$\sum_{\sigma \text{is the restriction of} \sigma'} \omega'(\sigma') = \omega(\sigma).$$

Then we finish the proof as in the previous case. ■

A *1-cell* in a matrix $B \in \mathbb{S}^{n \times m}$ is a submatrix B_{KL} such that $B_{ij} = 1$ for all $i \in K, j \in L$ and $B_{ij} \neq 1$ for all $i \in K, j \in \{1, \dots, \} \setminus L$ and $i \in \{1, \dots, m\} \setminus K, j \in L$.

Lemma 41 *Let $B(X) \in \mathbb{Z}[X]^{m \times n}$ be a symmetric matrix such that not all 1-entries of $B(X)$ are contained in 1-cells. Then* WCSP(B) *is #P-complete.*

Proof: If not all 1-entries of B are contained in 1-cells, then there are $i, j, k, \ell \in \{1, \dots, m\}$ such that $B_{ik} = B_{i\ell} = B_{jk} = 1$ and $B_{j\ell} \neq 1$. Let B' be the matrix obtained from B by replacing all entries different from 1 by 0. By Lemma 21, WCSP(B') is polynomial time reducible to WCSP(B). By Lemma 29, WCSP(B') is #P-complete.

Lemma 42 *Let $B \in \mathbb{Z}[X]^{m \times n}$ be a non-negative matrix whose entries are powers of the indeterminant X and let $C = (BB^\top)|_X$. Then for every 1-cell B_{KL} of B, the minor C_{KK} is a 1-cell of C.*

Proof: Let $C = (BB^\top)|_X$ and note that $C_{ij} = 1$ if and only if $B_{ik} = B_{jk} = 1$ for some $k \in \{1, \dots, n\}$. The claim follows. ■

$$\left(\begin{array}{c|c|c|c|c}
\begin{matrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1\end{matrix} & * & \cdots & * & * \\ \hline
* & \begin{matrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1\end{matrix} & \cdots & * & * \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
* & * & \cdots & \begin{matrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1\end{matrix} & * \\ \hline
* & * & \cdots & * & */1
\end{array}\right)$$

Figure 2: The cellular structure of the matrix B

By the results we have proved so far, from now on we may assume that our matrix B satisfies the following conditions. (This will be proved in Lemma 44.)

Conditions 43 *$B \in \mathbb{Z}[X]^{m \times m}$ such that:*

(1) $\mathsf{rank}(B) \geq 2$.

(2) B is symmetric.

(3) All entries of B are powers of the indeterminate X.
For $1 \leq i, j \leq n$, let $\ell_{ij} = \mathsf{deg}(B_{ij})$ (so $B_{ij} = X^{\ell_{ij}}$).

(4) There are a $k \geq 2$ and $1 = m_0 < m_1 < \ldots < m_k = m+1$ such that, for $1 \leq i \leq k-1$, the minors $B_{\{m_{i-1},\ldots,m_i-1\}}$ are 1-cells of B, the minor $B_{\{m_{k-1},\ldots,m_k-1\}}$ may be a 1-cell, and all 1-entries of B are contained in one of these 1-cells.

Condition 43 (4) means that B has a cellular structure as indicated in Figure 2. The $*$-cells contain no 1-entries; the bottom right cell is either a 1-cell or a $*$-cell. The 1-cells are squares on the diagonal, but they may be of different sizes.

Lemma 44 *Let $B \in \mathbb{S}^{m\times n}$ a non-negative matrix that has a block of rank at least 2. Then there is a matrix B' satisfying Conditions 43 such that* $\mathrm{WCSP}^+(B')$ *is polynomial time reducible to* $\mathrm{WCSP}(B)$.

Proof: We write $P \leq P'$ to denote that problem P is polynomial time reducible to problem P'.

By Corollary 26 and Lemma 27, there is a nonnegative matrix $B_1 \in \mathbb{Z}[X]^{m\times n}$ of rank at least 2 such that $\mathrm{WCSP}(B_1) \leq \mathrm{WCSP}(B)$. By Lemmas 32 and Lemma 34, there is a nonnegative symmetric matrix $B_2 \in \mathbb{Z}[X]^{m\times m}$ of rank at least 2 such that $\mathrm{WCSP}^+(B_2) \leq \mathrm{WCSP}(B_1)$. By Lemma 38, there is a positive symmetric matrix $B_3 \in \mathbb{Z}[X]^{m'\times m'}$ of rank at least 2 such that $\mathrm{WCSP}^+(B_3) \leq \mathrm{WCSP}^+(B_2)$. By Lemmas 23, Lemma 36, and Lemmas 37, there is a symmetric matrix $B_4 \in \mathbb{Z}[X]^{m'\times m'}$ of rank at least 2 such that all entries of B_4 are powers of X and $\mathrm{WCSP}^+(B_4) \leq \mathrm{WCSP}^+(B_3)$.

Note that B_4 satisfies conditions (1)–(3). By Lemma 39, we may assume that B_4 contains at least one 1-entry. If B_4 contains 1-entries that are not contained in some 1-cell, then $\mathrm{WCSP}(B_4)$ is #P-complete by Lemma 41, and thus we can reduce $\mathrm{WCSP}^+(B')$ for any matrix B' to $\mathrm{WCSP}^+(B_4)$. Thus we may assume that all 1-entries of B_4 are contained in 1-cells. If B_4 contains exactly one 1-cell, then this 1-cell must be a minor, because B_4 is symmetric. By permuting the rows and columns, we can bring B_4 into the desired form satisfying (4) with $k = 1$. If B_4 contains more than one 1-cell, then by Lemma 42, the matrix $B_5 = (B_4 B_4^\top)|_X$ satisfies (4) for some $k \geq 2$ (possibly after permuting rows and columns). Since B_5 contains two 1-cells, its rank is at least 2, thus it also satisfies (1). It immediately follows from the definition of B_5 that it satisfies (2) and (3). Finally, $\mathrm{WCSP}^+(B_5) \leq \mathrm{WCSP}^+(B_4)$ by Lemmas 34 and 36. ■

Thus it remains to prove the #P-completeness of $\mathrm{WCSP}^+(B)$ *for all matrices B satisfying Conditions 43.*

5.3 Matrices with at least two 1-cells

In this section, we will take care of those matrices B with at least two 1-cells. For such matrices, we may assume the following conditions in addition to Conditions 43:

Conditions 45 *(4) B has at least two 1-cells.*

(5) All diagonal entries of B are 1.

Lemma 46 *Let $B \in \mathbb{Z}[X]^{m\times m}$ be a symmetric positive matrix with at least two 1-cells. Then there is a matrix B' satisfying Conditions 43 and 45 such that* WCSP$^+(B')$ *is polynomial time reducible to* WCSP$^+(B)$.

Proof: Note that since B has two 1-cells, its rank is at least 2. By Lemmas 23, Lemma 36, and Lemmas 37, without loss of generality we may assume that all entries of B are powers of the indeterminate X. By Lemma 41, we may further assume that all 1-entries of B_4 are contained in 1-cells. Let $B_1 = (B \cdot B^\top)_X$, then WCSP$^+(B_1) \leq$ WCSP$^+(B)$, and possibly after permuting rows and columns, B_1 satisfies Conditions 43 and 45 (4). Condition 45 (5) can be achieved by Lemma 40. ■

Let B be a matrix satisfying Condition 43 (4). The *cells* of B are the submatrices B_{IJ}, where $I = \{n_{i-1}, \dots, n_i - 1\}$, $J = \{n_{j-1}, \dots, n_j - 1\}$ for some $1 \leq i, j \leq k$. These are precisely the "cells" of Figure 2. We call B a *cell matrix*, if for all cells B_{IJ} all entries are equal, that is, for $i, i' \in I$ and $j, j' \in J$ we have $B_{ij} = B_{i'j'}$.

Lemma 47 *Let $B(X) \in \mathbb{Z}^{m\times m}[X]$ be a matrix satisfying Conditions 43 and 45. Then there is a cell matrix $C(X) \in \mathbb{Z}^{m\times m}[X]$ that still satisfies Conditions 43 and 45, such that* WCSP$^+(C(X))$ *is polynomial time reducible to* WCSP$^+(B(X))$.

Proof: Observe that for every matrix $B(X)$ satisfying Conditions 43 and 45, the matrix $B'(X) = (B(X) \cdot B(X)^\top)|_X$ also satisfies the conditions, and WCSP$^+(B'(X))$ is polynomial time reducible to WCSP$^+(B(X))$. (We have already used this in the proof of Lemma 46.)

Furthermore, for $1 \leq i, j \leq m$ we have $B'_{ij}(X) = X^{n_{ij}}$, where

$$n_{ij} = \min_{1\leq k\leq m} \{\mathsf{deg}(B_{ik}(X)) + \mathsf{deg}(B_{jk}(X))\}$$

Since $B_{jj} = 1$ and thus $\mathsf{deg}(B_{jj}(X)) = 0$,

$$\mathsf{deg}(B'_{ij}(X)) \leq \mathsf{deg}(B_{ij}(X)).$$

Let $B_0(X) = B(X)$ and, for $i \geq 0$, $B_{i+1}(X) = (B_i(X) \cdot B_i(X)^\top)|_X$. Since the degrees of all entries are decreasing, there is a k such that $B_{k+1}(X) = B_k(X)$. We shall prove that $C(X) = B_k(X)$ is a cell matrix.

Let C_{IJ} be a cell of C that is not a 1-cell, and let $i \in I, j \in J$ such that $\mathsf{deg}(C(X)_{ij})$ is minimum among the degrees of all entries of the cell. Then, since $C(X) = C'(X) = (C(X) \cdot C(X)^\top)|_X$, for all $j' \in J$,

$$\begin{aligned}\mathsf{deg}(C_{ij'}(X)) &= \mathsf{deg}((C(X) \cdot C(X)^\top)|_X)_{ij}) \\ &= \min_{1\leq q\leq n} \{\mathsf{deg}(C_{iq}(X)) + \mathsf{deg}(C_{j'q}(X))\} \\ &\leq \mathsf{deg}(C_{ij}(X))\end{aligned}$$

because $C_{j'j}(X) = 1$. Thus by the minimality of $\mathsf{deg}(C_{ij}(X))$ we have $\mathsf{deg}(C_{ij}(X)) = \mathsf{deg}(C_{ij'}(X))$. Now for all $i' \in J$, analogously we get

$$\mathsf{deg}(C_{i'j'}(X)) \leq \mathsf{deg}(C_{ij'}(X)),$$

which implies $\deg(C_{ij}(X)) = \deg(C_{ij'}(X)) = \deg(C_{i'j'}(X))$. Thus $C_{ij}(X) = C_{i'j'}(X)$. ■

We now prove the #P-completeness of WCSP(B) for cell-matrices $B(X)$ satisfying Conditions 43 and 45 that have exactly two 1-cells.

Lemma 48 *Let $B(X) \in \mathbb{Z}[X]^{m\times m}$ be of the form*

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & 1 & X^\delta & \cdots & X^\delta \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & X^\delta & \cdots & X^\delta \\ \hline X^\delta & \cdots & X^\delta & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ X^\delta & \cdots & X^\delta & 1 & \cdots & 1 \end{array}\right).$$

Then the problem WCSP$(B(X))$ *is #P-complete.*

Proof: Let k, ℓ be the sizes of the two 1-cells in $B(X)$.

Since by Lemma 10, EVAL$((B(X)\cdot B(X)^\top)$ is polynomial time reducible to WCSP$(B(X))$, it suffices to prove that EVAL$((B(X)\cdot B(X)^\top)$ is #P-complete.

Analogously to the proof of Lemma 19 (also see [5]) it can be shown that for every symmetric matrix $A \in \mathbb{S}^{n\times n}$ the following graph version GCOUNT(A) of the problem COUNT(A) can be reduced to EVAL(A):

Input: Graph $G = (V, E)$, $w \in \mathbb{S}$.
Objective: Compute $N_A(G, w)$, the number of mappings $\sigma : V \to \{1, \dots, k\}$ with $\omega_A(\sigma) = w$.

This implies (as in Lemma 22) that EVAL$((B(X)\cdot B(X)^\top)|_X)$ is polynomial time reducible to EVAL$((B(X)\cdot B(X)^\top)$ and thus to WCSP(B). Observe that $(B(X)\cdot B(X)^\top)|_X = B(X)$. Let $B = B(2)$; we shall actually prove that EVAL(B) is #P-complete.

In [5], Dyer and Greenhill considered a generalised version of EVAL(A), in which vertex-weights are also allowed. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $F \in \mathbb{R}^{n\times n}$ a diagonal matrix with positive diagonal entries (the idea is that the entry F_{ii} denotes the weight assigned to i). For every graph $G = (V, E)$, let

$$Z_{A,F}(G) = \sum_{\sigma:V\to\{1,\dots,n\}} \prod_{\{u,v\}\in E} A_{\sigma(u)\sigma(v)} \prod_{v\in V} F_{\sigma(v)\sigma(v)}.$$

EVAL(A, F) is the problem of computing $Z_{A,F}(G)$ for a given graph $\mathcal{G}$. Dyer and Greenhill [5] proved that EVAL(A, F) is polynomial time reducible to EVAL(A).

Let $G = (V, E)$ be a graph. For every partition (V_1, V_2) of V, let $s(V_1, V_2) = |E \cap (V_1 \times V_2)|$ be the number of edges from V_1 to V_2. Observe that

$$Z_B(G) = \sum_{(V_1,V_2)\text{ Partition of }V} \ \sum_{\sigma_1:V_1\to\{1,\dots,\ell\}} \ \sum_{\sigma_2:V_2\to\{\ell+1,\dots,k\}} 2^{\delta\cdot s(V_1,V_2)}$$

$$= \sum_{(V_1,V_2) \text{ Partition of } V} k^{|V_1|} \ell^{|V_2|} 2^{\delta \cdot s(V_1,V_2)}$$
$$= Z_{A,F}(G),$$

where

$$A = \begin{pmatrix} 1 & 2^\delta \\ 2^\delta & 1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix}.$$

Thus EVAL(A, F) and therefore EVAL(A) is reducible to EVAL(D). It is easy to see that the #P-complete problem #MAX-CUT of counting the number of maximum cuts of a given graph is reducible to EVAL(A). ■

Lemma 49 *Let $B(X) \in \mathbb{Z}^{n\times n}[X]$ be a cell-matrix satisfying Conditions 43 and 45. Then* WCSP$^+(B(X))$ *is #P-complete.*

Proof: Let $\delta = \min\{\mathsf{deg}(B_{ij}(X)) \mid B_{ij}(X) \neq 1\}$ and $\Delta = \max\{\mathsf{deg}(B_{ij}(X)) \mid B_{ij}(X) \neq 1\}$.

Let B_{IJ} be a cell of B whose entries are X^δ. By symmetry and the definition of the cells, we have

$$C(X) = B_{(I\cup J)\,(I\cup J)}(X) = \left(\begin{array}{ccc|ccc} 1 & \dots & 1 & X^\delta & \cdots & X^\delta \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & X^\delta & \cdots & X^\delta \\ \hline X^\delta & \cdots & X^\delta & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ X^\delta & \dots & X^\delta & 1 & \cdots & 1 \end{array}\right).$$

By Lemma 48, WCSP$(C(X))$ is #P-complete. We shall reduce WCSP$(C(X))$ to WCSP$^+(B(X))$.

Let $\mathcal{D}' = (D', \alpha', \beta', \omega')$ be the canonical weighted template of $B(X)$. Let θ' be the diagonal of D' and, for $d \in D'$, $\kappa_d = \{d\}$. Recall that $D' = \{1, \dots, n\}^2$. Let $D = (I \cup J) \times (I \cup J) \subseteq D'$. $\alpha = \alpha' \cap D^2$, $\beta = \beta' \cap D^2$, and $\omega = \omega'|_D$. Observe that $\mathcal{D} = (D, \alpha, \beta, \omega)$ is isomorphic to the canonical weighted template for C. It will be more convenient to work with this template than with the canonical one.

Let $\mathcal{P} = (V; D; \mathcal{C})$ be an instance of WCSP(C). We define an instance $\mathcal{P}' = (V', D', \mathcal{C}')$ of WCSP$^+(B)$ as follows: Let $k = |V| \cdot \Delta + 1$, and let $i_0 \in I, j_0 \in J$.

(i) Let $V' = V \cup \{v^i_j \mid v \in V, 1 \leq i \leq 4, 1 \leq j \leq k\} \cup \{x, y\}$, where $k = |V| \cdot \Delta + 1$.

(ii) For every constraint $\langle (u, v), \alpha\rangle \in \mathcal{C}$, add the constraint $\langle (u, v), \alpha'\rangle$ to $\mathcal{C}'$. Similarly, for every constraint $\langle (u, v), \beta\rangle \in \mathcal{C}$, add the constraint $\langle (u, v), \beta'\rangle$ to $\mathcal{C}'$.

(iii) Add the constraints $\langle x, \kappa_{(i_0,j_0)}\rangle$ and $\langle y, \kappa_{(j_0,i_0)}\rangle$.

(iv) For every $v \in V$, $1 \leq i \leq 4$, and $1 \leq j < k$, add the constraints $\langle (v^i_j, v^i_{j+1}), \alpha \rangle$ and $\langle (v^i_j, v^i_{j+1}), \beta \rangle$.

(v) For every $v \in V$, add the constraints $\langle (x, v^1_1), \beta \rangle$, $\langle (x, v^2_1), \alpha \rangle$, $\langle (y, v^3_1), \beta \rangle$, $\langle (y, v^4_1), \alpha \rangle$.

(vi) For every $v \in V$, add the constraints $\langle (v, v^1_1), \alpha \rangle$, $\langle (v, v^2_1), \beta \rangle$, $\langle (v, v^3_1), \alpha \rangle$, $\langle (v, v^4_1), \beta \rangle$.

The constraints in (ii) make sure that the restriction of a solution of $\mathcal{P}'$ to V is a solution of $\mathcal{P}$, provided that the range of the solution of $\mathcal{P}'$ is contained in D. The constraints in (iii) guarantee that x is mapped to (i_0, j_0) and y is mapped to (j_0, i_0). The constraints in (iv) guarantee that v^i_j and $v^i_{j'}$ get the same value for all i, j, j'. The constraints in (v) force v^1_1 into column j_0, v^2_1 into row i_0, v^3_1 into column i_0, and v^4_1 into row j_0. Finally, the constraints in (vi) force v into the same row as v^1_1 and v^3_1, which also implies that v^1_1 and v^3_1 are forced in the same row. Moreover, they force v into the same column as v^2_1 and v^4_1.

For every solution δ' of $\mathcal{P}'$, let σ be the restriction of σ' to V. Observe that

$$\omega'(\sigma') = X^{2\delta} \cdot X^{K(\sigma')} \cdot \prod_{v \in V} \sigma(v)$$

for some K such that $0 \leq K(\sigma') \leq 4|V|\Delta$. The crucial observation is:

- If $\sigma'(v^i_1) \in D'$ for all $v \in V$ and $1 \leq i \leq 4$, then $K(\sigma') = 2k|V|\delta$.
- If $\sigma'(v^i_1) \in D \setminus D'$ for some $v \in V$ and $1 \leq i \leq 4$, then $K(\sigma') \geq 2k|V|\delta + k$.

Since $\mathsf{deg}(\prod_{v \in V} \sigma(v)) \leq |V| \cdot \Delta < k$ by the definition of k and since $\sigma'(v) \in D' \iff \sigma'(v^1_1), \ldots, \sigma'(v^4_1) \in D'$ (by the constraints in (vi)), it follows that

$$\mathsf{deg}(\omega'(\sigma')) < 2\delta + 2k|V|\delta + k \iff \sigma'(V) \subseteq D'.$$

This yields a reduction from WCSP$(C(X))$ to COUNT$^+(B(X))$ and thus to WCSP$^+(B(X))$ by Lemma 35. ∎

Corollary 50 *Let $B \in \mathbb{Z}[X]^{m \times m}$ be a symmetric positive matrix with at least two 1-cells. Then* WCSP$^+(B)$ *is #P-complete.*

5.4 Matrices with a single 1-cell

In this section we consider the remaining case that $B(X) \in \mathbb{Z}[X]^{m \times m}$ is matrix satisfying Conditions 43 and only has one 1-cell.

Before we get to the heart of the matter, we need two more simple reductions in the style of the previous sections.

Lemma 51 *Let $B(X) \in \mathbb{Z}^{m \times n}[X]$, $1 \leq s \leq m, 1 \leq t \leq m$, and let $C(X) \in \mathbb{Z}^{m \times n}[X]$ the matrix with*

$$C_{ij}(X) = B_{ij}(X) \cdot (B_{ii}(X))^s \cdot (B_{jj}(X))^t$$

for $1 \leq i \leq m, 1 \leq j \leq m$ Then WCSP$^+(C(X))$ *is polynomial time reducible to* WCSP$^+(B(X))$.

Proof: Note that the canonical templates for B and C only differ in their weight functions ω_B, ω_C. Let $\mathcal{P} = (V; D; \mathcal{C})$ be an instance of $\mathrm{WCSP}^+(C)$. We construct an instance $\mathcal{P}' = (V'; D; \mathcal{C}')$ of $\mathrm{WCSP}^+(B)$ as follows

(i) $V' = V \cup \{v_1^1, \ldots, v_s^1, v_1^2, \ldots, v_t^2 \mid v \in V\}$;

(ii) $\mathcal{C}' = \mathcal{C} \cup \mathcal{C}_1 \cup \mathcal{C}_2$ where:

- $\mathcal{C}_1 = \{\langle (v_1^1), \theta\rangle, \ldots, \langle (v_s^1), \theta\rangle, \langle (v_1^2), \theta\rangle, \ldots, \langle (v_t^2), \theta\rangle \mid v \in V\}$,
- $\mathcal{C}_2 = \{\langle (v, v_1^1), \alpha\rangle, \ldots, \langle (v, v_s^1), \alpha\rangle, \langle (v, v_1^2), \beta\rangle, \ldots, \langle (v, v_t^2), \beta\rangle \mid v \in V\}$.

Every solution σ of $\mathcal{P}$ can be extended to a solution of $\mathcal{P}'$ in a unique way, because, for any element $d \in D$, there is only one 'diagonal' element from the α-class and only one from β-class containing d. Conversely, the restriction of any solution of $\mathcal{P}'$ onto V is a solution of $\mathcal{P}$. Finally, for a solution of $\mathcal{P}$ and the corresponding solution σ' of $\mathcal{P}'$, we have

$$\begin{aligned}
\omega_B(\sigma') &= \prod_{v \in V} B_{\sigma_1(v)\sigma_2(v)} \cdot B^s_{\sigma_1(v)\sigma_1(v)} \cdot B^t_{\sigma_2(v)\sigma_2(v)} \\
&= \prod_{v \in V} C_{\sigma_1(v)\sigma_2(v)} \\
&= \omega_C(\sigma).
\end{aligned}$$

Here $\sigma(v) = (\sigma_1(v), \sigma_2(v))$ for all $v \in V$. ■

We can use the previous lemma to show that we may assume that our matrix B satisfies the following conditions (in addtions to Conditions 43).

Conditions 52 *Let $B \in \mathbb{Z}[X]^{m \times m}$ be a matrix satisfying Conditions 43, and, for $1 \leq i, j \leq m$, let $\ell_{ij} = \mathsf{deg}(B_{ij})$.*

(4) B has exactly one 1-cell.

Let r be the column of the first entry greater than 1 in row 1 of B, that is, $r = \min\{j \mid 1 \leq j \leq m, \ell_{1j} > 0\}$. Let also C be the matrix consisting of the first $r-1$ rows of B.

(5) $\mathsf{rank}(C) = 1$.

(6) For $1 \leq i < r$, $r \leq i' \leq m$, and $1 \leq j \leq m$,

$$\ell_{ij} \leq \ell_{i'j}.$$

Lemma 53 *Let $B \in \mathbb{Z}[X]^{m \times m}$ be a matrix satisfying Conditions 43 with precisely one 1-cell. Then there is a matrix B' satisfying Conditions 43 and 45 or Conditions 43 and 52 such that* $\mathrm{WCSP}^+(B')$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B)$.

Proof: We prove first that there is B' that either satisfies Conditions 43 and 45 or Conditions 43 and 52(5). We proceed by induction on m. If $m = 2$ then $r = 2$ and Conditions 52(5) holds trivially; so let $m \geq r > 2$ and assume that $\mathsf{rank}(C) > 2$. By Lemma 34 WCSP$^{+}(C \cdot C^{\top})$, is polynomial time reducible to WCSP$^{*}(C)$, which by Lemma 40 is polynomial time reducible to WCSP$^{*}(B)$ and hence to WCSP$^{+}(B)$. Moreover, $C \cdot C^{\top}$ is an $(r-1) \times (r-1)$-matrix with $\mathsf{rank}(C \cdot C^{\top}) = \mathsf{rank}(C) > 1$. If this matrix has at least two 1-cells, then the required B' can be obtained by Lemma 46. Otherwise, either Condition 52(5) holds for this matrix or we can apply the induction hypothesis.

Suppose now that B satisfies Condition 52(5). Choose $s \geq 0$ such that $\ell_{ij} \leq s\ell_{i'i'}$ for $1 \leq i, j \leq m$, $r \leq i' \leq m$. Let B' be the matrix obtained from B by applying Lemma 51 with s and $t = s$. For $1 \leq i, j \leq m$, let $\ell'_{ij} = \ell_{ij} + s \cdot \ell_{ii} + s \cdot \ell_{jj}$. Then $B'_{ij} = X^{\ell'_{ij}}$.

B' still satisfies Conditions 43 and 52(4),(5). We claim that it also satisfies 52(6). Let $1 \leq i < r$, $r \leq i' \leq m$, and $1 \leq j \leq m$. Then

$$\ell'_{ij} = \ell_{ij} + s\ell_{jj} \leq s\ell_{i'i'} + s\ell_{jj} \leq \ell_{i'j} + s\ell_{i'i'} + s\ell_{jj} = \ell'_{i'j}.$$

■

For the rest of this section, we fix a matrix $B \in \mathbb{Z}[X]^{m \times m}$ that satisfies Conditions 43 and 52. We also let $\ell_{ij} = \mathsf{deg}(B_{ij})$ for $1 \leq i, j \leq m$ and $r = \min\{j \mid 1 \leq j \leq m, \ell_{1j} > 0\}$.

Lemma 54 *Let k be a natural number and $B^{[k]}(X)$ the matrix with $B^{[k]}_{ij}(X) = B_{ij}(X) \cdot (B_{1j}(X))^{k-1}$. Then* WCSP$^{+}(B^{[k]}(X))$ *is polynomial time reducible to* WCSP$^{+}(B(X))$.

Proof: Let $\mathcal{D} = (D, \alpha, \beta, \omega)$ be the canonical weighted template for B and note that the canonical template for $B^{[k]}$ is the same except for its weight function, which we denote by ω_k.

Let $\mathcal{P} = (V; D; \mathcal{C})$ be an instance of WCSP$^{+}(B^{[k]})$. We construct an instance $\mathcal{P}' = (V'; D; \mathcal{C}')$ of WCSP$^{+}(B)$ as follows

(i) $V' = V \cup \{v_1, \dots, v_{k-1} \mid v \in V\} \cup \{x\}$;

(ii) Add all constraints in $\mathcal{C}$ to $\mathcal{C}'$.

(iii) For every $v \in V$, add the constraints $\langle (v_1, x), \alpha \rangle, \dots, \langle (v_{k-1}, x), \alpha \rangle$ to $\mathcal{C}$.

(iv) For every $v \in V$, add the constraints $\langle (v, v_1), \beta \rangle, \dots, \langle (v, v_{k-1}), \beta \rangle$.

The constraints in (ii) guarantee that the restriction of every solution of $\mathcal{P}'$ to V is a solution of $\mathcal{P}$. The constraints in (iii) make sure that all v_i are mapped to the first row. The constraints in (iv) make sure that for every v the v_i are mapped to the same column as v. Thus (iii) and (iv) to gether imply that if v is mapped to (i, j) by a solution, then $v_1, \dots, v_{k-1}$ are mapped to $(1, j)$.

Thus every solution σ of $\mathcal{P}$ can be extended to a solution σ' of $\mathcal{P}'$ in a unique way, and conversely, the restriction σ of any solution σ' of $\mathcal{P}'$ to V is a solution of $\mathcal{P}$. Furthermore, for every solution σ' of $\mathcal{P}'$,

$$\omega(\sigma') = B_{11} \cdot \prod_{v \in V} B_{\sigma_1(v)\sigma_2(v)} \cdot B^{k-1}_{1\sigma_2(v)}$$

$$= B_{11} \cdot \prod_{v \in V} C_{\sigma_1(v)\sigma_2(v)}$$
$$= B_{11} \cdot \omega_k(\sigma).$$

Here $\sigma(v) \in (\sigma_1(v), \sigma_2(v))$. ■

In the following, for $k \geq 1$, we let

$$C^{[k]}(X) = B^{[k]}(X) \cdot B^{[k]}(X)^{\top}. \tag{5}$$

Observe that $C^{[k]}(X)$ is a symmetric positive matrix in $\mathbb{Z}[X]^{m \times m}$.

Let f be a polynomial and λ a complex number. Then $\mathsf{mult}(\lambda, f)$ denotes the *multiplicity* of λ in f if λ is a root of f and $\mathsf{mult}(\lambda, f) = 0$ otherwise. We shall use the following basic facts on polynomials, roots and their multiplicity.

Lemma 55 *Let f be a polynomial and λ a complex number.*

(1) There exists a unique (up to a scalar factor) irreducible polynomial p_λ such that λ is a root of p_λ. If λ is a root of f then $p_\lambda | f$.

(2) If $\mathsf{mult}(\lambda, f) = s$ then $f = p_\lambda^s \overline{f}$ and $\overline{f}(\lambda) \neq 0$.

(3) For every root λ of $f(X^k)$, λ^k is a root of $f(X)$. Moreover, $\mathsf{mult}(\lambda, f(X^k)) = \mathsf{mult}(\lambda^k, f(X))$.

For every root λ of $C_{11}^{[1]}$, every $r \leq j \leq m$ and k, we denote the multiplicity of $\lambda^{1/k}$ in $C_{1j}^{[k]}$ by $m(\lambda, j, k)$, and we let

$$m(\lambda, j) = \min_{k \geq 1} m(\lambda, j, k).$$

Lemma 56 *(1) For any root λ of $C_{11}^{[1]}$, any $r \leq j \leq m$ and any positive integer k, $\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda, C_{11}^{[1]}) \geq m(\lambda, j)$ and $\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq m(\lambda, j)$.*

(2) For any root λ of $C_{11}^{[1]}$, any $r \leq j \leq m$ such that the first and jth rows are linearly dependent and any positive integer k, $\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]})$.

(3) For any $r \leq j \leq m$ such that the first and the jth row are linearly independent, there is a root λ of $C_{11}^{[1]}$ such that $\mathsf{mult}(\lambda, C_{11}^{[1]}) > m(\lambda, j)$

Proof: Let $j \in \{r, \dots, m\}$. Let $b = \min\{\ell_{j1} - \ell_{11}, \dots, \ell_{jm} - \ell_{1m}\}$. By Conditions 52(6), $b \geq 0$.

To simplify the notation, let $a_1 = \ell_{11} = 0, \dots, a_{r-1} = \ell_{1r-1} = 0$, $a_r = \ell_{1r}, \dots, a_m = l_{1m}$, $b_1 = \ell_{j1} - b, \dots, b_m = \ell_{jm} - b$ and $c_i = b_i - a_i$ for $1 \leq i \leq m$. Note that $c_i \geq 0$

for $1 \le i \le m$ and all the c_i are equal 0 if and only if the first and jth rows are linearly dependent. Then

$$
\begin{aligned}
B &= \begin{pmatrix} X^{a_1} & X^{a_2} & X^{a_3} & \dots & X^{a_m} \\ \vdots & \vdots & \vdots & & \vdots \\ X^{b+b_1} & X^{b+b_2} & X^{b+b_3} & \dots & X^{b+b_m} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}, \\
B^{[k]} &= \begin{pmatrix} X^{ka_1} & X^{ka_2} & X^{ka_3} & \dots & X^{ka_m} \\ \vdots & \vdots & \vdots & & \vdots \\ X^{b+\overbrace{b_1-a_1}^{=c_1}+ka_1} & X^{b+\overbrace{b_2-a_2}^{=c_2}+ka_2} & X^{b+\overbrace{b_3-a_3}^{=c_3}+ka_3} & \dots & X^{b+\overbrace{b_m-a_m}^{=c_m}+ka_m} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}.
\end{aligned}
$$

For the matrix $C^{[k]}$ we have

$$
\begin{aligned}
C_{11}^{[k]} &= X^{2ka_1} + X^{2ka_2} + \ldots + X^{2ka_m}, & (6) \\
C_{1j}^{[k]} &= X^{b}(X^{c_1+2ka_1} + X^{c_2+2ka_2} + \ldots + X^{c_m+2ka_m}), & (7) \\
C_{jj}^{[k]} &= X^{2b}(X^{2c_1+2ka_1} + X^{2c_2+2ka_2} + \ldots + X^{2c_m+2ka_m}). & (8)
\end{aligned}
$$

Take a root λ of $C_{11}^{[1]}$. Then $\lambda \neq 0$, because $a_1 = 0$. Let $k \ge 1$. Note first that (6) and Lemma 55(3) imply that $\lambda^{1/k}$ is a root of $C_{11}^{[k]}$ with

$$
\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda, C_{11}^{[1]}). \tag{9}
$$

If the first and jth rows are linearly dependent, then $c_1 = c_2 = \ldots = c_m$. Thus by Equalities (6)–(8),

$$
C_{1j}^{[k]} = X^{b+c_1} \cdot C_{11}^{[k]} \quad \text{and} \quad C_{jj}^{[k]} = X^{2b+2c_1} \cdot C_{11}^{[k]}.
$$

Since $\lambda \neq 0$, it follows that

$$
\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}).
$$

This proves Lemma 56(2).

In the following, we assume that rows 1 and j are linearly independent. In particular, not all the c_i are equal. If, for some k, $\lambda^{1/k}$ is not a root of $C_{1j}^{[k]}$, then $m(\lambda, j) = 0$, and Lemma 56(1) and (3) hold trivially. In the following, we assume that $\lambda^{1/k}$ is a root of $C_{1j}^{[k]}$. Our first goal is to find $m(\lambda, j)$.

Let $\alpha \in \mathbb{C}$ such that $\lambda = e^{\alpha}$. Then for every $k \ge 1$,

$$
C_{1j}^{[k]}(\lambda^{1/k}) = e^{\alpha b/k}\left(e^{\alpha(2a_1+\frac{c_1}{k})} + e^{\alpha(2a_2+\frac{c_2}{k})} + \ldots + e^{\alpha(2a_m+\frac{c_m}{k})}\right) = 0
$$

(because $\lambda^{1/k}$ is a root of $C_{1j}^{[k]}$). Consider the function

$$f_\lambda(z) = e^{\alpha(2a_1+c_1z)} + e^{\alpha(2a_2+c_2z)} + \ldots + e^{\alpha(2a_m+c_mz)}.$$

For every $k \geq 1$ we have $f_\lambda(1/k) = 0$.

CLAIM 1. Suppose that $g(z) = u(z)+iv(z)$ is a function analytic in the real segment $[0,1]$ and there are sequences $\{r_n\}$, $\{s_n\}$ from the real segment $[0,1]$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = 0$ and $u(r_n) = v(s_n) = 0$. Then

(a) $g(0) = 0$;

(b) there are sequences $\{r'_n\}$, $\{s'_n\}$ from the real segment $[0,1]$ such that

$$\lim_{n\to\infty} r'_n = \lim_{n\to\infty} s'_n = 0$$

and $u'(r'_n) = v'(s'_n) = 0$ where u', v' denote the derivatives of the corresponding functions.

Proof: Without loss of generality we may assume that $\{r_n\}$, $\{s_n\}$ are monotone. Then, since g is continuous,

$$g(0) = \lim_{z\to 0} g(z) = \lim_{z\to 0} u(z) + i \lim_{z\to 0} v(z) = \lim_{n\to\infty} u(r_n) + i \lim_{n\to\infty} v(s_n) = 0.$$

Furthermore, let u_0, v_0 denote the restrictions of u, v onto the real interval $[0,1]$. Then u_0, v_0 are continuous and differentiable real functions. Therefore, for any n, there are $r'_n \in [r_{n+1}, r_n]$ and $s'_n \in [s_{n+1}, s_n]$ such that $u'_0(r'_n) = v'_0(s'_n) = 0$. Clearly, $\lim_{n\to\infty} r'_n = \lim_{n\to\infty} s'_n = 0$ and $u'(r'_n) = u'_0(r'_n) = 0$, $v'(s'_n) = v'_0(s'_n) = 0$.

This completes the proof of Claim 1.

The function $f_\lambda(z)$ is analytic everywhere including $[0,1]$. Moreover, for any k,

$$f_\lambda(1/k) = e^{\alpha(2a_1+\frac{c_1}{k})} + e^{\alpha(2a_2+\frac{c_2}{k})} + \ldots + e^{\alpha(2a_m+\frac{c_m}{k})} = \frac{C_{1j}^{[k]}(\lambda^{1/k})}{e^{b/k}} = 0.$$

Therefore, by Claim 1, for any $\ell \geq 1$, the ℓth derivative $f_\lambda^{(\ell)}(0) = 0$.

Computing the derivatives at 0 we get

$$f_\lambda^{(l)}(0) = (\alpha c_1)^l e^{2\alpha a_1} + (\alpha c_2)^l e^{2\alpha a_2} + \ldots + (\alpha c_m)^l e^{2\alpha a_m} = 0.$$

Observe that for $1 \leq i \leq r-1$ we have $c_i = c_1$, because by Condition 52(5), the first r rows of our matrix B are linearly dependent and hence identical, because their first entry is 1. Therefore, without loss of generality we may assume that

$$c_1 = \ldots = c_{s_1}, \quad c_{s_1+1} = \ldots = c_{s_2}, \quad \ldots, \quad c_{s_{t-1}+1} = \ldots = c_{s_t} = 0,$$

where $s_0 = 0, s_t = m$, and that $c_{s_1}, , \dots , , c_{s_t}$ are all different. Moreover, we have $t \geq 2$, because not all the c_i are equal by our assumption that rows 1 and j are linearly independent, and $s_1 \geq r - 1$.

Denoting $Y_i = e^{2\alpha a_{s_{i-1}+1}} + \dots + e^{2\alpha a_{s_i}}$, $1 \leq i \leq t-1$, we get a system of linear equations

$$\begin{cases} c_{s_1}Y_1 + c_{s_2}Y_2 + \dots + c_{s_{t-1}}Y_{t-1} &= 0 \\ (c_{s_1})^2Y_1 + (c_{s_2})^2Y_2 + \dots + (c_{s_{t-1}})^2Y_{t-1} &= 0 \\ &\vdots \\ (c_{s_1})^{t-1}Y_1 + (c_{s_2})^{t-1}Y_2 + \dots + (c_{s_{t-1}})^{t-1}Y_{t-1} &= 0 \end{cases}$$

The determinant of the system is Vandermonde. Therefore, $Y_1 = \dots = Y_{t-1} = 0$. Denoting

$$\begin{aligned} g_1(X) &= X^{2a_1} + \dots + X^{2a_{s_1}} \\ &\vdots \\ g_{t-1}(X) &= X^{2a_{s_{t-2}+1}} + \dots + X^{2a_{s_{t-1}}} \\ g_t(X) &= X^{2a_{s_{t-1}+1}} + \dots + X^{2a_{s_t}}, \end{aligned}$$

we have $g_1(\lambda) = \dots = g_{t-1}(\lambda) = 0$ and, since $C_{11}^{[1]}(X) = g_1(X) + \dots + g_{t-1}(X) + g_t(X)$ and λ is a root of $C_{11}^{[1]}$, $g_t(\lambda) = 0$ as well.

Everything we have done so far is independent of the specific root λ. Thus, for every irreducible polynomial g with $g|C_{11}^{[1]}$, we have $g|g_1, \dots , g_t$. Let $h_1, \dots , h_q$ be the different irreducible divisors of $C_{11}^{[1]}$. Without loss of generality we may assume that the leading coefficients of the h_i are positive. Then

$$\begin{aligned} C_{11}^{[1]}(X) &= \overline{g}_1(X)h_1^{r_{11}}(X)\dots h_q^{r_{1q}}(X) + \dots + \overline{g}_t(X)h_1^{r_{t1}}(X)\dots h_q^{r_{tq}}(X) \\ &= h_1^{m_1}(X)\dots h_q^{m_q}(X) \\ &\quad \cdot \big(\overline{g}_1(X)h_1^{r'_{11}}(X)\dots h_q^{r'_{1q}}(X) + \dots + \overline{g}_t(X)h_1^{r'_{t1}}(X)\dots h_q^{r'_{tq}}(X)\big), \end{aligned}$$

for suitably chosen polynomials $\overline{g}_i(X)$ and non-negative integers r_{ij}, r'_{ij}, and $m_i = \min(r_{1i}, \dots , r_{ti})$. To simplify the notation, we set

$$h(X) = h_1^{m_1}(X)\dots h_q^{m_q}(X), \qquad f_i(X) = \overline{g}_i(X)h_1^{r'_{i1}}(X)\dots h_q^{r'_{iq}}(X).$$

Then $g_i(X) = h(X) \cdot f_i(X)$ for $1 \leq i \leq t$ and

$$C_{11}^{[1]}(X) = h(X) \cdot \big(f_1(X) + \dots + f_t(X)\big).$$

Since $s_1 \geq r - 1$, the polynomial g_1 is the only one with a non-zero constant term. Thus g_1 and g_i for $2 \leq i \leq t$ differ by more than a constant factor. Since $h(X)$ is the greatest common divisor of $g_1, \dots , g_t$, the degree of at least one of the polynomials f_i is positive. Let $1 \leq i \leq t$. Since all coefficients of g_i and the leading coefficient of $h(X)$ are positive, the leading coefficient of f_i is positive. Thus

$$\mathsf{deg}(f_1(X) + \dots + f_t(X)) > 0. \tag{10}$$

To simplify the notation in Claims 2 and 3, suppose now that λ is a root of h_1.

CLAIM 2. $m(\lambda, j) = m_1$.
Proof: We need to show that $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) \geq m_1$ for all k and that there is a positive integer k such that $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}(X)) = m_1$.

As is easily seen, for any k,

$$\begin{aligned} C_{1j}^{[k]}(X) &= X^b(X^{c_{s_1}} g_1(X^k) + \ldots + X^{c_{s_t}} g_t(X^k)) \\ &= X^b h(X^k)\big(X^{c_{s_1}} f_1(X^k) + \ldots + X^{c_{s_t}} f_t(X^k)\big). \end{aligned} \tag{11}$$

Therefore $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) \geq \mathsf{mult}(\lambda, h) = m_1$.

Regroup the summands in $f_\lambda(z)$:

$$\begin{aligned} f_\lambda(z) &= e^{\alpha c_{s_1} z}(e^{2\alpha a_1} + \ldots + e^{2\alpha a_{s_1}}) + \ldots + e^{\alpha c_{s_t} z}(e^{2\alpha a_{s_{t-1}+1}} + \ldots + e^{2\alpha a_{s_t}}) \\ &= h(\lambda)\big(\lambda^{c_{s_1} z} f_1(\lambda) + \ldots + \lambda^{c_{s_t} z} f_t(\lambda)\big). \end{aligned}$$

Let

$$\bar{f}(X, z) = X^{c_{s_1} z} f_1(X) + \ldots + X^{c_{s_t} z} f_t(X).$$

Then $f_\lambda(z) = h(\lambda) \cdot \bar{f}(\lambda, z)$.

Let $\beta_1 = \lambda^{c_{s_1}/t!}, \ldots, \beta_t = \lambda^{c_{s_t}/t!}$. Then, for any $\ell \leq t$,

$$f_\lambda\left(\frac{\ell}{t!}\right) = h(\lambda) \cdot \bar{f}\left(\lambda, \frac{\ell}{t!}\right) = h(\lambda) \cdot \big(\beta_1^\ell f_1(\lambda) + \ldots + \beta_t^\ell f_t(\lambda)\big).$$

Suppose for contradiction that $\bar{f}(\lambda, \ell/t!) = 0$ for $l = 1, \ldots, t$. Consider the system

$$\begin{cases} \beta_1 f_1(\lambda) + \ldots + \beta_t f_t(\lambda) &= 0 \\ (\beta_1)^2 f_1(\lambda) + \ldots + (\beta_t)^2 f_t(\lambda) &= 0 \\ & \vdots \\ (\beta_1)^t f_1(\lambda) + \ldots + (\beta_t)^t f_t(\lambda) &= 0 \end{cases}$$

Since $\beta_i \neq \beta_{i'}$ whenever $i \neq i'$, we get $f_1(\lambda) = \ldots = f_t(\lambda) = 0$, which contradicts (10).

Thus for some $\ell \leq t$, $\bar{f}(\lambda, \ell/t!) \neq 0$. Pick such an ℓ and let $k = t!/\ell$. Note that

$$\bar{f}\left(X^k, 1/k\right) = X^{c_{s_1}} f_1(X^k) + \ldots + X^{c_{s_t}} f_t(X^k)$$

and recall (11). Since $\bar{f}(\lambda, 1/k) \neq 0$, $\lambda^{1/k}$ is not a root of the polynomial on the left hand side, and by (11) this implies $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) = m_1$.

This completes the proof of Claim 2.

CLAIM 3. For every every positive integer k, $\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq m_1$.

Proof: Let us consider $C_{jj}^{[k]}$:

$$\begin{aligned} C_{jj}^{[k]}(X) &= X^{2b}(X^{2c_1+2ka_1}+\ldots+X^{2c_m+2ka_m}) \\ &= X^{2b}(X^{2c_1}(X^k)^{2a_1}+\ldots+X^{2c_m}(X^k)^{2a_m}) \\ &= X^{2b}h(X^k)(X^{2c_{s_1}}f_1(X^k)+\ldots+X^{2c_{s_t}}f_t(X^k)). \end{aligned}$$

Then

$$\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq \mathsf{mult}(\lambda^{1/k}, h(X^k)) = \mathsf{mult}(\lambda, h(X)) = m_1.$$

This completes the proof of Claim 3.

Clearly, Lemma 56(1) follows from Claims 1 and 2. To prove (3), we recall that

$$C_{11}^{[1]}(X) = h_1^{m_1}(X)\ldots h_q^{m_q}(X)\big(f_1(X)+\ldots+f_t(X)\big).$$

Thus every root λ of $f_1(X)+\ldots+f_t(X)$ is also a root of $C_{11}^{[1]}(X)$ and, therefore, it is a root of one of $h_1,\ldots,h_q$. Then $\mathsf{mult}(\lambda, C_{11}^{[1]}) > m(\lambda,j)$. Choose k by Claim 2. Then

$$\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda, C_{11}^{[1]}) > m_i = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}).$$

Note that such a number k exists for every root λ such that $\mathsf{mult}(\lambda, C_{11}^{[1]}) > m(\lambda,j)$. ∎

Lemma 57 *There exist $j \in \{r,\ldots,m\}$, a root λ of $C_{11}^{[k]}$ and a positive integer k such that*

(1) $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) < \mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]})$;

(2) for every $i \in \{r,\ldots,m\}$, $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) \leq \mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]})$.

Proof: We choose λ and $j \in \{r,\ldots,m\}$ such that the first and jth rows of $B(X)$ are linearly inpendent and $m(\lambda,j)$ is the least number for all pairs λ,j satisfying Lemma 56(3). By Lemma 56(1), $\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq m(\lambda,j) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]})$ for a certain k. For any $i \neq j$, if $m(\lambda,i) \leq m(\lambda,i)$ and the first and ith rows are linearly independent, then the pair λ,i satisfies Lemma 56(3), $m(\lambda,i) = m(\lambda,j)$ by the choice of λ,j and $\mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]}) \geq m(\lambda,i) = m(\lambda,j) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]})$. If $m(\lambda,i) \geq m(\lambda,j)$ and the first and ith rows are linearly independent, then

$$\mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]}) \geq m(\lambda,i) \geq m(\lambda,j) = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}).$$

Finally, if the first and ith rows are linearly dependent, then, by Lemma 56(2),

$$\mathsf{mult}(\lambda^{1/k}, C_{1i}^{[k]}) = \mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) > m(\lambda,j)$$

and

$$\mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]}) \geq \mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) > m(\lambda,j).$$

■

Finally, we are ready to put everything together.

Proof (of Theorem 16): By Lemma 44, without loss of generality, we may assume that the matrix B satisfies Conditions 43, and we have to prove that WCSP$^+(B)$ is #P-complete. If B has at least two 1-cells, then WCSP$^+(B(X))$ is #P-complete by Corollary 50.

So we assume that B has exactly one 1-cell and satisfies Conditions 43 and 52 (the latter by Lemma 53). We use the same notation as above; in particular, we define r as in Condition 52(4) and $C^{[k]}$ as in (5) on page 36. We proceed by induction on r. As in the base case of induction, $r = 2$, our argument is the same as for induction step, we prove them simultaneously.

Choose j, λ, and k according to Lemma 57 and let

$$t = \mathsf{mult}(\lambda^{1/k}, C^{[k]}_{1j}).$$

Then $\mathsf{mult}(\lambda^{1/k}, C^{[k]}_{11}) > t$ and $\mathsf{mult}(\lambda^{1/k}, C^{[k]}_{ii}) \geq t$ for $r \leq i \leq m$. Let p be an irreducible polynomial such that λ is a root of p and let $D(Y)$ denote the matrix with entries

$$(D(Y))_{ij} = Y^{\ell}, \qquad \text{where } \ell \geq 0 \text{ such that } (C^{[k]}_{ij})|_p = p(X)^{\ell}.$$

By Lemmas 36 and 37, WCSP$^+(D)$ is polynomial time reducible to WCSP$^+(B)$.

Case 1. For any u, v, $\mathsf{mult}(\lambda^{1/k}, C^{[k]}_{uv}) \geq t$.

In this case, the matrix C obtained from D by dividing by X^t has at least two 1-cells, since $C_{1j} = C_{j1} = 1$, but $C_{11} \neq 1$. Then WCSP$^+(C(X))$ is #P-complete by Corollary 50.

Case 2. There is an entry $C^{[k]}_{uv}$ such that $\mathsf{mult}(\lambda^{1/k}, C^{[k]}_{uv}) < t$.

In this case, let s be the least multiplicity of $\lambda^{1/k}$ in the entries of $C^{[k]}$. Denote by C the matrix D divided by p^s.

Subcase 2.1. u, v can be chosen such that $u \geq r$ or $v \geq r$.

Then $C_{uv} = C_{vu} = 1$, $C_{uu} \neq 1$ or $C_{vv} \neq 1$. Therefore uth and vth rows of C are linearly independent and C has at least two 1-cells. Then WCSP$^+(C(X))$ is #P-complete by Corollary 50.

Subcase 2.2. $1 \leq u, v < r$ whenever $\mathsf{mult}(\lambda^{1/k}, C^{[k]}_{uv}) = s$.

If C has at least two 1-cells then WCSP$^+(C(X))$ is #P-complete as above. If C has only one 1-cell, then the size of this 1-cell is less than $r - 1$, because $\mathsf{mult}(\lambda^{1/k}, C^{[k]}_{11}) > t > s$. As is easily seen, C satisfies Conditions 43; by Lemma 53, it can be assumed to satisfy Conditions 52. Since, by Lemma 37, WCSP$^+(C)$ is polynomial time reducible to WCSP$^+(D(X))$, we can apply induction hypothesis.

Finally, notice that Subcase 2.2 is impossible in the base case of the induction, $r = 2$.

■

6 Conclusions

We give a complete complexity theoretic classification for the problem of evaluating the partition function of a symmetric non-negative matrix A, which may be viewed as the adjacency matrix of an undirected weighted graph H. Our proofs explore a correspondence between this evaluation problem and weighted constraint satisfaction problems for constraint languages with two equivalence relations.

Peculiarly, our proof does not go through for matrices with negative entries. Indeed, we do not know whether the evaluation problem for the matrix

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

is #P-complete. (Observe that the evaluation problem for this matrix is equivalent to the problem of counting induced subgraphs with an even number of edges.)

The more important open problem is to obtain a classification result for the evaluation problem for non-symmetric matrices, corresponding to directed graphs. We believe that with our results such a classification may now be within reach, in particular because our main hardness result goes through for directed graphs. The ultimate goal of this line of research is a classification of counting and weighted CSP for arbitrary constraint languages. Towards a solution of this problem, one may try to reduce the weighted CSP to evaluation problems for directed graphs. It is interesting to note that the known reduction between the corresponding decision problems does not give a reduction between the counting problems we are interested in here.

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