Programming Research Group

CONSTRAINT SATISFACTION PROBLEMS AND FINITE ALGEBRAS

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Abstract

Many natural combinatorial problems can be expressed as constraint satisfaction problems. This class of problems is known to be **NP**-complete in general, but certain restrictions on the form of the constraints can ensure tractability. In this paper we show that any restricted set of constraint types can be associated with a finite universal algebra. We explore how the computational complexity of a restricted constraint satisfaction problem is connected to properties of the corresponding algebra. For this, we introduce a notion of 'tractable algebra' and study how the tractability of an algebra relates to the tractability of its smaller derived algebras, including its subalgebras and homomorphic images. This allows us to significantly reduce the types of algebras which need to be investigated. Using these results we exhibit a common structural property of all known intractable constraint satisfaction problems. Finally, we classify all finite strictly simple surjective algebras with respect to tractability. The result is a dichotomy theorem which significantly generalises Schaefer's dichotomy for the Generalised Satisfiability problem.

1 Introduction

The constraint satisfaction problem provides a framework in which it is possible to express, in a natural way, a wide variety of combinatorial problems [2, 16, 18]. The aim in a constraint satisfaction problem is to find an assignment of values to a given set of variables, subject to constraints on the values which can be assigned simultaneously to certain specified subsets of the variables.

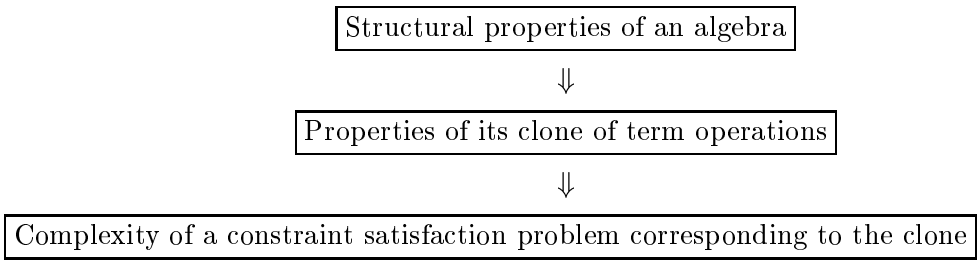
The mathematical framework used to describe constraint satisfaction problems has strong links with several other areas of computer science and mathematics. For example, links with relational database theory [5] and with some notions of group theory [3] and universal algebra [11] have been investigated. There is a survey of these results in [20].

The constraint satisfaction problem is known to be **NP**-complete in general [15]. However, certain restrictions on the form of the constraints can ensure tractability [3, 8, 9, 12, 13]. In [12], it is proved that the time complexity of a problem involving a restricted set of constraint relations is determined by certain algebraic invariance properties of the constraint relations, namely, by the clone of polymorphisms of these relations. This result transforms the search for new tractable constraint types to the search for clones leading to tractability.

Clone theory (for a general introduction see [21, 24]) is a part of modern universal algebra. The branch of mathematics called universal algebra deals with general structures such as algebras (i.e. sets endowed with families of operations), varieties (i.e. classes of algebras closed under certain operators), relational structures etc. It is natural to expect that more universal-algebraic techniques may be involved in the study of constraint satisfaction problems.

In the present paper we consider constraint satisfaction problems where all of the constraint relations belong to the finite power subuniverse system of a certain finite algebra. To any algebra one can assign, in a natural way, a clone, the so-called clone of term operations. It is a well-known phenomenon in universal algebra that there are strong links between the structural properties of an algebra and properties of its term operations.

The best example demonstrating the power of this theory is the book [6]. Another example is the Baker-Pixley theorem [1] that provides a precise algebraic characterisation of the constraint types for which local consistency ensures global consistency [10]. The focus of this paper is to examine how the structural properties of an algebra affect the complexity of the corresponding constraint satisfaction problem.



The paper is organised as follows. In Section 2 we give basic definitions including the definition of tractable algebra, and establish the fundamental connection between constraint satisfaction problems, clones of operations and finite algebras. We show that Schaefer's theorem [23] concerning the complexity of the generalised satisfiability problem can be viewed as a classification of two-element algebras with respect to tractability. In Section 3 we prove that, in this context, it suffices to consider idempotent algebras, that is, algebras whose unary term operations are trivial. Further on, we study how the tractability of a finite algebra relates to the tractability of its smaller derived algebras. We also discuss some problems arising in our investigation. In Section 4 we use the results obtained to classify all strictly simple surjective algebras. We show that such algebras give rise to constraint satisfaction problems which are either tractable or **NP**-complete. Thus we extend Schaefer's dichotomy theorem to a much larger class of problems.

2 Definitions

2.1 Constraint Satisfaction Problems

The central notion in the study of constraints and constraint satisfaction problems is the notion of a *relation*.

Definition 2.1 *For any set A, and any natural number n, the set of all n-tuples of elements of A is denoted by A^n. Any subset ρ of A^n is called an n-ary* relation *on A. By R_A we denote the set of all finitary relations on A.*

We now define the standard constraint satisfaction problem which has been widely studied [2, 13, 14, 15, 16, 18].

Definition 2.2 *The* constraint satisfaction problem (**CSP**) *is the combinatorial decision problem with*

Instance: *a triple $(V, A, \mathcal{C})$ where*

- *V is a set of* variables*;*
- *A is a domain of* values*;*
- *$\mathcal{C}$ is a set of* constraints*, $\{C_1, \ldots, C_q\}$.*

  Each constraint $C_i \in \mathcal{C}$ is a pair $\langle s_i, \rho_i \rangle$, where
  - *s_i is a tuple of variables of length m_i, called the* constraint scope*;*
  - *ρ_i is an m_i-ary relation on A, called the* constraint relation*.*

Question: *does there exist a* solution*, i.e. a function f, from V to A, such that, for each constraint $\langle s_i, \rho_i \rangle \in \mathcal{C}$, with $s_i = (x_{i_1}, \ldots, x_{i_m})$, the tuple $(f(x_{i_1}), \ldots, f(x_{i_m}))$ belongs to ρ_i?*

Example 2.3 An instance of the standard propositional SATISFIABILITY problem [4, 19] is specified by giving a formula in propositional logic, that is, a conjunction of clauses, and asking whether there are values for the variables which make the formula true.

Suppose that $\phi = X_1 \wedge \cdots \wedge X_n$ is such a formula, where the X_i are clauses. The satisfiability question for ϕ can be expressed as the instance $(V, \{0,1\}, \mathcal{C})$ of **CSP**, where V is the set of all variables appearing in the clauses X_i, and $\mathcal{C} = \{\langle s_1, \rho_1 \rangle, \ldots, \langle s_n, \rho_n \rangle\}$. The constraints $\langle s_k, \rho_k \rangle$ are constructed as follows.

For every clause X_k, where $x_1^k, \ldots, x_{j_k}^k$ are all variables appearing in X_k, let $s_k = (x_1^k, \ldots, x_{j_k}^k)$ and $\rho_k = \{0,1\}^{j_k} \setminus \{(a_1, \ldots, a_{j_k})\}$ where $a_i = 0$ if x_i^k is negated and $a_i = 1$ otherwise (i.e., ρ_k consists of all j_k-tuples that make X_k true).

It is easy to see that solutions of this **CSP** instance are exactly assignments which make the formula ϕ true. Hence, any instance of SATISFIABILITY can be expressed as an instance of **CSP**.

Example 2.4 An instance of GRAPH UNREACHABILITY consists of a graph $G = (V, E)$ and a pair of vertices, $v, w \in V$. The question is whether there is no path in G from v to w. This can expressed as the **CSP** instance $(V, \{0,1\}, \mathcal{C})$ where

$$\mathcal{C} = \{\langle e, \{(0,0),(1,1)\}\rangle | e \in E\} \cup \{\langle (v), \{(0)\}\rangle, \langle (w), \{(1)\}\rangle\}.$$

A number of other examples of well-known problems expressed as **CSP**s can be found in [11].

The general constraint satisfaction problem is known to be **NP**-complete (as seen from Example 2.3). However, certain restrictions may affect the complexity of **CSP**. One of the possible natural ways to restrict **CSP** is to limit the relations which can appear in constraints.

Definition 2.5 *Let Γ be a subset of R_A. Denote by* **CSP**(Γ) *the subclass of* **CSP** *defined by the following property: any constraint relation in any instance must belong to Γ.*

For $A = \{0,1\}$, the time complexity of **CSP**(Γ) was completely classified by Schaefer [23]. In particular, he proved that such a problem is either tractable (i.e., solvable in

polynomial time) or **NP**-complete. The dichotomy problem for larger domains is still open and seems to be very interesting and hard [3].

The following two examples are of particular interest for us, as will be seen in Section 3.

Example 2.6 It can easily be checked that Not-All-Equal Satisfiability [4, 19] coincides with the problem $\mathbf{CSP}(\{N\})$ where N is the following ternary relation on $\{0,1\}$:

$$N = \{(a,b,c) \mid \{a,b,c\} = \{0,1\}\}.$$

Example 2.7 An instance of Graph q-Colorability consists of a graph G. The question is whether the vertices of G can be labelled with q colours so that adjacent vertices are assigned different colours.

This problem corresponds to $\mathbf{CSP}(\{\neq_B\})$ where B is a q-element set (of colours) and

$$\neq_B \ = \{(a,b) \in B^2 \mid a \neq b\}$$

is the disequality relation on B.

It is well known (see [4, 19]) that this problem belongs to **P** if $q \leq 2$, and is **NP**-complete otherwise.

2.2 Operations and Clones

By an n-ary operation on a set A we mean any mapping $f : A^n \to A$. The set of all finitary operations on A is denoted by O_A. In this paper we shall need some special operations.

- An operation of the form $e^i_n(x_1, \ldots, x_n) = x_i$ is called a *projection*.
- An operation f on A is called *idempotent* if $f(x, \ldots, x) = x$ for any $x \in A$.
- A binary operation $f(x,y)$ is called a *semilattice* operation[1], if it satisfies the following three conditions:
  - $f(x, f(y,z)) = f(f(x,y),z)$ (Associativity)
  - $f(x,y) = f(y,x)$ (Commutativity)
  - $f(x,x) = x$ (Idempotency)
- An n-ary operation f is said to be a *near-unanimity* operation if $f(y,x,\ldots,x) = f(x,y,x,\ldots,x) \cdots f(x,\ldots,x,y) = x$ for any $x, y \in A$. Ternary near-unanimity operations are called *majority* operations.
- A ternary operation f is said to be *affine* if $f(x,y,z) = x - y + z$ for any $x, y, z \in A$, where $(A, +, -)$ is an Abelian group.

[1] Note that in earlier papers [9, 12, 20] the term *ACI operation* is used

- A *dual discriminator* is a ternary operation defined as

$$d(x,y,z) = \begin{cases} y, & \text{if } y = z, \\ x, & \text{otherwise} \end{cases}$$

  Note that d is a majority operation.

Definition 2.8 ([24]) *A set $C \subseteq O_A$ is called a* clone *on A if*

- *C is closed under composition;*
- *C contains all projections.*

Suppose $f(x_1, \ldots, x_n) \in O_A$ and let M be an $m \times n$ matrix whose entries a_{ij} are elements from A. Set

$$f(M) = \begin{pmatrix} f(a_{11}, \ldots, a_{1n}) \\ \vdots \\ f(a_{m1}, \ldots, a_{mn}) \end{pmatrix}.$$

Definition 2.9 ([21, 24]) *An n-ary operation $f \in O_A$* preserves *an m-ary relation $\rho \in R_A$ (or f is a* polymorphism *of ρ, or ρ is* invariant *under f) if, for any $m \times n$ matrix M whose columns belong to ρ, the column $f(M)$ belongs to ρ as well.*

For given $F \subseteq O_A$ and $\Gamma \subseteq R_A$, let

$$Inv(F) = \{\rho \in R_A \mid \rho \text{ is invariant under each operation from } F\},$$

$$Pol(\Gamma) = \{f \in O_A \mid f \text{ is a polymorphism of each relation from } \Gamma\}.$$

Proposition 2.10 ([21]) *Let A be a finite set. For any $\Gamma_1, \Gamma_2 \subseteq R_A$, and for any $F_1, F_2 \subseteq O_A$, the following holds:*
(i) if $\Gamma_1 \subseteq \Gamma_2$ then $Pol(\Gamma_2) \subseteq Pol(\Gamma_1)$,
(ii) if $F_1 \subseteq F_2$ then $Inv(F_2) \subseteq Inv(F_1)$,
(iii) $Inv(Pol(\Gamma_1)) \supseteq \Gamma_1$,
(iv) $Pol(Inv(F_1)) \supseteq F_1$.

Proposition 2.10 states that the mappings $Pol(-)$ and $Inv(-)$ induce the so-called Galois connection between O_A and R_A. This connection with application to **CSP**s is discussed in detail in [11].

The following theorem says that any clone is a clone of polymorphisms.

Theorem 2.11 ([21]) *Let A be a finite set. A set $C \subseteq O_A$ is a clone on A if and only if $C = Pol(Inv(C))$.*

Sets of the form $Inv(F)$ are sometimes called *relational clones*. As follows from Proposition 2.10, They are in one-to-one correspondence with clones of operations. Relational clones can be characterised as sets of relations closed under some natural operations (see e.g. [21]). Using this characterisation Jeavons has proved a nice theorem connecting the complexity of $\mathbf{CSP}(\Gamma)$ and the clone of operations $Pol(\Gamma)$.

Theorem 2.12 ([12]) *Let A be a finite set, and $\Gamma, \Gamma_0 \subseteq R_A$. If Γ_0 is finite and $Pol(\Gamma) \subseteq Pol(\Gamma_0)$ then $\mathbf{CSP}(\Gamma_0)$ is reducible in polynomial time to $\mathbf{CSP}(\Gamma)$.*

Informally speaking, Theorem 2.12 says that the complexity of $\mathbf{CSP}(\Gamma)$ is determined by the clone $Pol(\Gamma)$. This theorem is discussed in more detail in [20]. It is shown in [12] that Schaefer's result [23] concerning the complexity of generalised satisfiability problems can easily be proved using Theorem 2.12 and the classical description of clones on a two-element set [22].

Corollary 2.13 *Let A be a finite set, $\Gamma, \Gamma_0 \subseteq R_A$. If Γ_0 is finite and $Inv(Pol(\Gamma)) \supseteq Inv(Pol(\Gamma_0))$ then $\mathbf{CSP}(\Gamma_0)$ is reducible in polynomial time to $\mathbf{CSP}(\Gamma)$.*

Corollary 2.13 says that searching for tractable sets of relations one may restrict oneself to the study of sets of the form $Inv(F)$.

Definition 2.14 *A set $\Gamma \subseteq R_A$ is said to be* tractable *if $\mathbf{CSP}(\Gamma)$ is tractable.*

It was proved in [9, 20] that $Inv(\{f\})$ is tractable if f is a constant operation, or a semilattice operation, or an affine operation, or a near-unanimity operation.

2.3 Algebras

In this subsection we give basic definitions of algebraic constructions. In our definitions we mainly follow [17] and [24].

Definition 2.15 *An* algebra *is an ordered pair $\mathcal{A} = (A, F)$ such that A is a nonempty set and F is a family of finitary operations on A The set A is called the* universe *of $\mathcal{A}$, the operations from F are called* basic. *An algebra with finite universe is referred to as a* finite algebra.

Definition 2.16 *Let $\mathcal{A} = (A, F)$ be an algebra. The n-*th direct power *of $\mathcal{A}$ is the algebra $\mathcal{A}^n = (A^n, F)$ where we treat every operation $f_i \in F$ as acting on A^n component-wise, i.e.*

$$f_i((a_{11}, \ldots, a_{1n}), \ldots, (a_{n_i 1}, \ldots, a_{n_i n})) = (f_i(a_{11}, \ldots, a_{n_i 1}), \ldots, f_i(a_{1n}, \ldots, a_{n_i n})).$$

Definition 2.17 *Let $\mathcal{A} = (A, F)$ be an algebra and B a subset of A such that, for any $f_i \in F$, and for any $b_1, \ldots, b_{n_i} \in B$, we have $f_i(b_1, \ldots, b_{n_i}) \in B$. Then the algebra $\mathcal{B} = (B, F|_B)$, where $F|_B$ consists of restrictions of operations f_i to B, is called a* subalgebra *of $\mathcal{A}$. If $B \neq A$ then $\mathcal{B}$ is said to be* proper.

Definition 2.18 *A* finite power subuniverse system *of an algebra $\mathcal{A}$ is the set $SP_{\mathrm{fin}}(\mathcal{A}) = \{\rho \mid \rho$ is a universe of a subalgebra of $\mathcal{A}^n$ for some $n \geq 1\}$.*

Remark 2.19 *$SP_{\mathrm{fin}}(\mathcal{A}) = Inv(F) \subseteq R_A$.*

Remark 2.19 identifies the fact that relational clones and finite power subuniverse systems of finite algebras are, in fact, the same things viewed differently.

Definition 2.20 *Let $\mathcal{A} = (A, F)$ be an algebra. The least clone on A containing F is called the* clone of term operations *of $\mathcal{A}$ and is denoted by $Clo(\mathcal{A})$. Two algebras with the same universe are called* term equivalent *if they have the same clone of term operations.*

In other words, the clone $Clo(\mathcal{A})$ consists of all operations that can be obtained from F and the projections by means of composition.

Theorem 2.21 ([24]) *For a finite algebra $\mathcal{A}$, $SP_{\text{fin}}(\mathcal{A})$ is the greatest set Γ with $Pol(\Gamma) = Clo(\mathcal{A})$.*

Corollary 2.22 *If finite algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are term equivalent then the equality $SP_{\text{fin}}(\mathcal{A}_1) = SP_{\text{fin}}(\mathcal{A}_2)$. holds.*

Main Definition. *We call an algebra $\mathcal{A}$* (globally) tractable [**NP**-complete] *if $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$ is tractable [**NP**-complete], and* locally tractable[2] *if, for every finite $\Gamma \subseteq SP_{\text{fin}}(\mathcal{A})$, $\mathbf{CSP}(\Gamma)$ is tractable.*

Problem 2.23 *Characterise all finite tractable algebras.*

Given a set of relations $\Gamma \subseteq R_A$, one can consider the algebra $\mathcal{A}_\Gamma = (A, Pol(\Gamma))$. According to Proposition 2.10 and Theorems 2.11 and 2.21, the algebra $\mathcal{A}_\Gamma$, or any algebra term equvalent to $\mathcal{A}_\Gamma$, is the richest algebra (i.e. with the largest clone of term operations) among those containing Γ in their finite power subuniverse systems. Clearly, tractabilty of $\mathcal{A}_\Gamma$ is equivalent to tractability of $\mathbf{CSP}(\Gamma)$. In this way, Schaefer's result [23] may be interpreted as a classification of two-element algebras with respect to tractability.

Theorem 2.24 (Schaefer) *A two-element algebra $\mathcal{A} = (\{0,1\}, F)$ is* **NP**-*complete if it is term equivalent to an algebra $(\{0,1\}, G)$ where G is a permutation group on $\{0,1\}$. Otherwise $\mathcal{A}$ is tractable.*

The central idea in the remainder of this paper is to generalise this result by using algebraic properties of $\mathcal{A}_\Gamma$ in order to obtain information about the clone $Pol(\Gamma)$ whose properties, as Theorem 2.12 indicates, are crucial for tractability of $\mathbf{CSP}(\Gamma)$.

[2]We must distinguish the notions of global tractability and local tractability because there is no guarantee that the following situation cannot happen: for any finite subset Γ' of some infinite set, Γ, of relations, there exists a particular p[olynomial algorithm $Alg(\Gamma')$ solving $\mathbf{CSP}(\Gamma')$, but there is no uniform p[olynomial algorithm solving any instance of $\mathbf{CSP}(\Gamma)$.

3 Tractability and Algebraic Constructions

Throughout the rest of the paper we consider only finite algebras.

In this section we first show that, when studying the tractability of finite algebras we can restrict our attention to certain special classes of algebras.

Definition 3.1 *We call an algebra* surjective[3] *if all of its term operations are surjective.*

It is easy to verify that a finite algebra is surjective if and only if its unary term operations form a permutation group.

Definition 3.2 ([26]) *Let $\mathcal{A} = (A, F)$ be an algebra, and U a non-empty subset of A. The* induced term algebra $\mathcal{A}|_U$ *is defined as $(U, (Clo\mathcal{A})|_U)$ where $(Clo\mathcal{A})|_U = \{g|_U : g \in Clo\mathcal{A}$ and g preserves $U\}$.*

It is easy to check that $Clo(\mathcal{A}|_U) = (Clo\mathcal{A})|_U$.

Theorem 3.3 *For any finite algebra $\mathcal{A}$ there exists a set U such that*
i) $\mathcal{A}|_U$ is surjective;
ii) If $\mathcal{A}$ is locally tractable then so is $\mathcal{A}|_U$;
iii) If $\mathcal{A}|_U$ is (locally) tractable then so is $\mathcal{A}$.

Proof. Let $\mathcal{A} = (A, F)$ be a finite algebra, and let U be a minimal subset of A which is the range of some unary operation $f \in Clo\mathcal{A}$.

Clearly, $\mathcal{A}|_U$ is surjective by the choice of U. We now prove that it satisfies properties (ii) and (iii).

(ii) Let Γ be a finite subset of $SP_{\text{fin}}(\mathcal{A}|_U)$. We will show that $\mathbf{CSP}(\Gamma)$ is linear-time reducible to $\mathbf{CSP}(\Gamma')$ where Γ' is a suitable finite subset of $SP_{\text{fin}}(\mathcal{A})$.

Let $\mathcal{P} = (V, U, \mathcal{C})$ be an instance of $\mathbf{CSP}(\Gamma)$. For any relation $\rho \in \Gamma$, with arity m, we denote by ρ' the universe of the subalgebra of $\mathcal{A}^m$ generated by ρ, i.e.,

$$\rho' = \{g(\overline{a_1}, \ldots, \overline{a_k}) \mid k > 0, g \in Clo(\mathcal{A}) \text{ and } \overline{a_1}, \ldots, \overline{a_k} \in \rho\}.$$

Let $\Gamma' = \{\rho' \mid rho \in \Gamma\}$ and set $\mathcal{P}' = (V, A, \mathcal{C}')$ where $\mathcal{C}' = \{\langle s, \rho' \rangle \mid \langle s, \rho \rangle \in \mathcal{C}\}$. It is clear that $\mathcal{P}'$ is a problem instance of $\mathbf{CSP}(\Gamma')$ and that it can be obtained from $\mathcal{P}$ in linear time. Since all the projections belong to $Clo(\mathcal{A})$, we have $\rho \subseteq \rho'$, so any solution of $\mathcal{P}$ is also a solution of $\mathcal{P}'$. Now, suppose ψ is a solution of $\mathcal{P}'$. We shall prove that $f\psi$ is a solution of $\mathcal{P}$. For this, we show that $f(\rho') \subseteq \rho$. Let $\overline{a} \in \rho'$. Then $f(\overline{a}) = fg(\overline{a_1}, \ldots, \overline{a_k})$ for some $g \in Clo(\mathcal{A})$ and some $\overline{a_1}, \ldots, \overline{a_k} \in \rho$. The operation $fg(x_1, \ldots, x_k)$ preserves U and belongs to $Clo(\mathcal{A})$; therefore its restriction to U belongs to $Clo(\mathcal{A})|_U$. Since $\rho \in SP_{\text{fin}}(\mathcal{A}|_U)$, we get $f(\overline{a}) \in \rho$.

(iii) We shall show that $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$ can be reduced in linear time to $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}|_U))$. First note that, by the choice of U, $f|_U$ is a permutation, so, by iterating as necessary, we obtain a unary operation $f' \in Clo\mathcal{A}$ such that $f'(a) = a$ for all a in U

[3]Note that in [25] an algebra is said to be surjective if all of its basic operations are surjective. Such algebras can have non-surjective term operations.

Let $\mathcal{P} = (V, A, \mathcal{C})$ be an instance of $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$, where $\mathcal{C} = \{\langle s_1, \rho_1\rangle, \ldots, \langle s_q, \rho_q\rangle\}$, and consider the triple $\mathcal{P}' = \langle V, U, \mathcal{C}'\rangle$, where $\mathcal{C}' = \{\langle s_1, f'(\rho_1)\rangle, \ldots, \langle s_q, f'(\rho_q)\rangle\}$ and $f'(\rho_i) = \{f'(\overline{a}) \mid \overline{a} \in \rho_i\}$.

We first prove that $\mathcal{P}'$ is an instance of $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}|_U))$. Since the range of f' is U, any relation of the form $f'(\rho_i)$ can be viewed as a relation on U. We need to show that, for $1 \leq i \leq q$, $f'(\rho_i)$ belongs to $SP_{\text{fin}}(\mathcal{A}|_U)$, i.e. for any $g(x_1, \ldots, x_m) \in Clo(\mathcal{A}|_U)$, and for any $\overline{a_1}, \ldots, \overline{a_m} \in f'(\rho_i)$, we have $\overline{a} = g(\overline{a_1}, \ldots, \overline{a_m}) \in f'(\rho_i)$. The operation g is the restriction of some $g' \in Clo\mathcal{A}$ to U such that g' preserves U. Suppose that $\overline{a_j} = f'(\overline{b_j})$ for some $\overline{b_j} \in \rho_i$, $1 \leq j \leq m$. Then

$$\overline{a} = g(\overline{a_1}, \ldots, \overline{a_m}) = g'(f'(\overline{b_1}), \ldots, f'(\overline{b_m})).$$

We have $\overline{a} \in \rho_i$, because all elements $\overline{b_j}$ belong to ρ_i, and $g'(f'(x_1), \ldots, f'(x_m)) \in Clo(\mathcal{A})$. All components of $\overline{a}$ belong to U, since g' preserves U. Finally, we have $\overline{a} = f'(\overline{a}) \in f'(\rho_i)$, since $f'(a) = a$ for any $a \in U$.

Since each $\rho_i \in \mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$, we have $f'(\rho_i) \subseteq \rho_i$; therefore any solution of $\mathcal{P}'$ is also a solution of $\mathcal{P}$. Furthermore, if ψ is a solution of $\mathcal{P}$ then $f'\psi$ is a solution of $\mathcal{P}'$. Clearly, $\mathcal{P}'$ can be obtained from $\mathcal{P}$ in linear time. Therefore tractability of $\mathcal{A}|_U$ implies that $\mathcal{A}$ is tractable. Finally, local tractability of $\mathcal{A}|_U$ implies local tractability of $\mathcal{A}$, since if all the constraint relations in $\mathcal{C}$ are taken from some finite set $\Gamma \subseteq SP_{\text{fin}}(\mathcal{A})$ then every constraint relation in $\mathcal{C}'$ belongs to a finite subset $\{f'(\rho) \mid \rho \in \Gamma\}$ of $SP_{\text{fin}}(\mathcal{A}|_U)$. □

The next theorem shows that we need only consider surjective algebras which are idempotent. To establish this result we need two auxiliary lemmas from [24].

Definition 3.4 *The* full idempotent reduct *of an algebra $\mathcal{A} = (A, F)$ is the algebra $(A, Clo_{id}(\mathcal{A}))$ where $Clo_{id}(\mathcal{A})$ consists of all idempotent operations from $Clo(\mathcal{A})$.*

For an n-ary relation ρ on A, and a number $1 \leq i \leq n$, let

$$\text{pr}_i\rho = \{b_i \in A \mid (b_1, \ldots, b_n) \in \rho\}.$$

Lemma 3.5 *Let $\mathcal{A} = (A, F)$ be a finite algebra and $\mathcal{A}_0$ its full idempotent reduct. Then, for any n-ary relation $\rho \in SP_{\text{fin}}(\mathcal{A}_0)$ such that $|\text{pr}_i\rho| > 1$ for all $1 \leq i \leq n$, there exist $l \geq 0$, an $(n+l)$-ary relation $\sigma_\rho \in SP_{\text{fin}}(\mathcal{A})$, and elements $b_1, \ldots, b_l \in A$ such that*

$$(x_1, \ldots, x_n) \in \rho \leftrightarrow (x_1, \ldots, x_n, b_1, \ldots, b_l) \in \sigma_\rho.$$

Proof. Follows immediately from Proposition 1.11 in [24]. □

Let G be a permutation group on a finite set $A = \{a_1, \ldots, a_k\}$. The relation ρ_G is defined as follows:

$$\rho_G = \{(g(a_1), \ldots, g(a_k)) \mid g \in G\}.$$

Lemma 3.6 *Let $\mathcal{A}$ be a finite algebra whose unary term operations form a permutation group G. Then $\rho_G \in SP_{\text{fin}}(\mathcal{A})$.*

Proof. Follows from Proposition 1.3 in [24]. □

Theorem 3.7 *A finite surjective algebra $\mathcal{A}$ is locally tractable if and only if its full idempotent reduct is locally tractable.*

Proof. Let $\mathcal{A}_0$ denote the full idempotent reduct of $\mathcal{A}$. We have $Clo(\mathcal{A}_0) = Clo_{id}(\mathcal{A}) \subseteq Clo(\mathcal{A})$. It follows from Proposition 2.10 that $SP_{\text{fin}}(\mathcal{A}) \subseteq SP_{\text{fin}}(\mathcal{A}_0)$, so $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$ can be reduced to $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}_0))$ in constant time.

Now let Γ be a finite subset of $SP_{\text{fin}}(\mathcal{A}_0)$. We shall show that $\mathbf{CSP}(\Gamma)$ can be reduced in polynomial time to $\mathbf{CSP}(\Gamma')$, where Γ' is a finite subset of $SP_{\text{fin}}(\mathcal{A})$.

Let $\mathcal{P} = (V, A, \mathcal{C})$ be an instance of $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}_0))$. First do the following procedure:

- While there is a constraint $\langle s, \rho\rangle \in \mathcal{C}$ with $s = (x_{i_1}, \ldots, x_{i_n})$, and a number j such that $|\text{pr}_j\rho| = 1$, say $\text{pr}_j\rho = \{a\}$, do
  - for each constraint $\langle s^0, \rho^0\rangle \in \mathcal{C}$ such that s contains x_{i_j}, at some position k, do
    - remove x_{i_j} from s^0;
    - replace ρ^0 by the relation

$$\{(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \mid (a_1, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_n) \in \rho\}.$$

(Note that this relation is still a member of $SP_{\text{fin}}(\mathcal{A}_0)$.)

It is clear that this procedure can be carried out in polynomial time. The result of carrying out this procedure is a new instance $\mathcal{P}' = (V', A, \mathcal{C}')$ of $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}_0))$ such that $V' \subseteq V$ and no constraint relation ρ in $\mathcal{C}$ has a coordinate position j with $|\text{pr}_j\rho| = 1$. It can straightforwardly be verified that $\mathcal{P}'$ has solutions if and only if $\mathcal{P}$ has solutions.

For each element $a \in A$, introduce a variable y_a such that $y_a \notin V'$ and let $V'' = V' \cup \{y_a \mid a \in A\}$. Suppose $\langle s', \rho'\rangle$ is a constraint in $\mathcal{C}'$ where ρ' is of arity n. Then there exist a number $l \geq 0$, a $(n+l)$-ary relation $\sigma_{\rho'} \in SP_{\text{fin}}(\mathcal{A})$, and elements $b_1, \ldots, b_l \in A$ with the properties described in Lemma 3.5. If $s' = (x_{j_1}, \ldots, x_{j_n})$ then let $s'' = (x_{j_1}, \ldots, x_{j_n}, y_{b_1}, \ldots, y_{b_l})$. Now let

$$\mathcal{C}'' = \{\langle s'', \sigma_{\rho'}\rangle \mid \langle s', \rho'\rangle \in \mathcal{C}'\} \cup \{\langle (y_{a_1}, \ldots, y_{a_k}), \rho_G\rangle\}.$$

Consider the instance $\mathcal{P}'' = (V'', A, \mathcal{C}'')$ of $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$. It can be obtained from $\mathcal{P}'$ in linear time. Assume that ψ is a solution of $\mathcal{P}'$. Let

$$\psi'(x) = \begin{cases} \psi(x), & \text{if } x \in V' \\ a, & \text{if } x = y_a \end{cases}$$

It is easy to check that ψ' is a solution of $\mathcal{P}''$.

Now let ψ be a solution of $\mathcal{P}''$. Then there exists a permutation $g \in G$ such that $(\psi(y_{a_1}), \ldots, \psi(y_{a_k})) = (g(a_1), \ldots, g(a_k))$. Denote by ψ' the restriction of ψ to V'. We shall now prove that $g^{-1}\psi'$ is a solution of $\mathcal{P}'$, where $g^{-1} \in G$ is the permutation inverse

to g. Note that any relation from $SP_{\rm fin}(\mathcal{A})$ is invariant under g^{-1}, because g^{-1} is a term operation of $\mathcal{A}$.

Suppose $\langle s', \rho' \rangle$ is a constraint in $\mathcal{C}'$, and $s' = (x_{j_1}, \dots, x_{j_n})$. Then we have

$$\begin{aligned} s' = (x_{j_1}, \dots, x_{j_n}) \Longrightarrow s'' = (x_{j_1}, \dots, x_{j_n}, y_{b_1}, \dots, y_{b_l}) \Longrightarrow \\ (\psi(x_{j_1}), \dots, \psi(x_{j_n}), \psi(y_{b_1}), \dots, \psi(y_{b_l})) \in \sigma_{\rho'} \Longleftrightarrow \\ (\psi'(x_{j_1}), \dots, \psi'(x_{j_n}), g(b_1), \dots, g(b_l)) \in \sigma_{\rho'} \Longleftrightarrow \\ (g^{-1}\psi'(x_{j_1}), \dots, g^{-1}\psi'(x_{j_n}), g^{-1}g(b_1), \dots, g^{-1}g(b_l)) \in \sigma_{\rho'} \Longleftrightarrow \\ (g^{-1}\psi'(x_{j_1}), \dots, g^{-1}\psi'(x_{j_n}), b_1, \dots, b_l) \in \sigma_{\rho'} \Longleftrightarrow \\ (g^{-1}\psi'(x_{j_1}), \dots, g^{-1}\psi'(x_{j_n})) \in \rho'. \end{aligned}$$

Since there exist only finitely many relations ρ' which can be obtained from the relations from Γ applying the procedure described above, any relation $\sigma_{\rho'}$ can be constructed from ρ in polynomial time. So, $\mathcal{P}''$ can be obtained from $\mathcal{P}$ in polynomial time, and $\mathcal{P}''$ has a solution if and only if $\mathcal{P}$ has a solution. □

The next two theorems connect the tractability of a finite algebra with the tractability of its subalgebras and *homomorphic images*.

Definition 3.8 *Let $\mathcal{A}_1 = (A_1, F_1)$ and $\mathcal{A}_2 = (A_2, F_2)$ such that $F_1 = (f_i^1 \mid i \in I)$ and $F_2 = (f_i^2 \mid i \in I)$. A map $\varphi : A_1 \to A_2$ is called a* homomorphism *from $\mathcal{A}_1$ to $\mathcal{A}_2$ if $\varphi f_i^1(a_1, \dots, a_{n_i}) = f_i^2(\varphi(a_1), \dots, \varphi(a_{n_i}))$ holds for all $i \in I$ and all $a_1, \dots, a_{n_i} \in A_1$. If the map φ is onto then $\mathcal{A}_2$ is said to be a* homomorphic image *of $\mathcal{A}_1$. A homomorphic image of a subalgebra of an algebra $\mathcal{A}$ is called a* factor *of $\mathcal{A}$.*

Theorem 3.9 *Let $\mathcal{A}$ be a finite algebra. Then*

(i) if $\mathcal{A}$ is (locally) tractable then so is every subalgebra of $\mathcal{A}$.

(ii) if $\mathcal{A}$ has an **NP***-complete subalgebra then $\mathcal{A}$ is* **NP***-complete itself.*

Proof. Let $\mathcal{A}'$ be a subalgebra of $\mathcal{A}$. It is easy to check that $SP_{\rm fin}(\mathcal{A}') \subseteq SP_{\rm fin}(\mathcal{A})$. Hence, $\mathbf{CSP}(SP_{\rm fin}(\mathcal{A}'))$ can be reduced to $\mathbf{CSP}(SP_{\rm fin}(\mathcal{A}))$ in constant time.

Now (i) and (ii) follow from the above consideration. □

Theorem 3.10 *If a finite algebra $\mathcal{A}$ is locally tractable then so is every homomorphic image of $\mathcal{A}$.*

Proof. Let $\mathcal{B}$ be a homomorphic image of $\mathcal{A}$ and let φ be the corresponding homomorphism. We will show that, for any finite $\Gamma \subseteq SP_{\rm fin}(\mathcal{B})$, $\mathbf{CSP}(\Gamma)$ is linear-time reducible to $\mathbf{CSP}(\Gamma')$ for some finite $\Gamma' \subseteq SP_{\rm fin}(\mathcal{A})$.

For $\rho \in SP_{\rm fin}(\mathcal{B})$, set $\varphi^{-1}(\rho) = \{\overline{a} \mid varphi(\overline{a}) \in \rho\}$ where φ acts component-wise. It is clear that $\varphi^{-1}(\rho)$ is a relation of the same arity as ρ. It can straightforwardly be checked that $\varphi^{-1}(\rho) \in SP_{\rm fin}(\mathcal{A})$. Let $\Gamma' = \{\varphi^{-1}(\rho) \mid rho \in \Gamma\}$. Then Γ' is a finite subset of $SP_{\rm fin}(\mathcal{A})$.

Take an instance $\mathcal{P} = (V, B, \mathcal{C})$ of $\mathbf{CSP}(\Gamma)$ and construct an instance $\mathcal{P}' = (V, A, \mathcal{C}')$ of $\mathbf{CSP}(\Gamma')$ where $\mathcal{C}' = \{\langle s, \varphi^{-1}(\rho)\rangle \mid \langle s, \rho\rangle \in \mathcal{C}\}$.

If ψ is a solution of $\mathcal{P}'$ then $\varphi\psi$ is a solution of $\mathcal{P}$. Conversely, if ξ is a solution of $\mathcal{P}$ then any function $\psi : V \to A$ such that $\varphi\psi(v) = \xi(v)$ for any $v \in V$ is a solution of $\mathcal{P}'$. □

Corollary 3.11 *If $\mathcal{A}$ is a locally tractable finite algebra then so is every factor of $\mathcal{A}$.*

The preceding corollary shows that a locally tractable finite algebra has no intractable subalgebras or homomorphic images. It is not known whether tractability of all proper subalgebras of a finite algebra $\mathcal{A}$ suffices for tractability of $\mathcal{A}$. However, the next example shows that the tractability of all homomorphic images of a finite algebra $\mathcal{A}$ does not ensure the tractability of $\mathcal{A}$,

Example 3.12 Consider the algebra $\mathcal{A} = (\{0,1,2\}, \circ)$ where $\circ$ is a binary operation defined by the following Cayley table[4]

$\circ$	0	1	2
0	0	1	1
1	0	1	1
2	1	1	2

We will show that $\mathcal{A}$ is **NP**-complete even though all of its smaller homomorphic images are tractable.

The algebra $\mathcal{A}$ has a subalgebra $\mathcal{A}' = (\{0,1\}, \circ)$ such that $x \circ y = y$ for any $x, y \in \{0,1\}$. It is easy to see that the relation N (see Example 2.6) belongs to $SP_{\text{fin}}(\mathcal{A}')$. Therefore the **NP**-complete problem NOT-ALL-EQUAL SATISFIABILITY [4] is reducible in polynomial time to $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}'))$. It follows from Theorem 3.9 that $\mathcal{A}$ is **NP**-complete.

Now we prove that all smaller homomorphic images of $\mathcal{A}$ are tractable. Since one-element algebras are certainly tractable, we need to consider only two-element homomorphic images $\mathcal{B}$ of $\mathcal{A}$. Let $\mathcal{B} = (\{a,b\}, *)$ where $*$ is a binary operation on $\{a,b\}$ and assume that φ is a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then we have $\varphi(x \circ y) = \varphi(x) * \varphi(y)$ for any $x, y \in \{0,1,2\}$.

Case 1. $\varphi(0) = \varphi(1) = a, \varphi(2) = b$.

We have

$$\begin{aligned} a * a &= \varphi(0) * \varphi(0) = \varphi(0 \circ 0) = \varphi(0) = a, \\ a * b &= \varphi(0) * \varphi(2) = \varphi(0 \circ 2) = \varphi(1) = a, \\ b * a &= \varphi(2) * \varphi(0) = \varphi(2 \circ 0) = \varphi(1) = a, \\ b * b &= \varphi(2) * \varphi(2) = \varphi(2 \circ 2) = \varphi(2) = b. \end{aligned}$$

It is easy to see that $*$ is a semilattice operation on $\{a,b\}$. Any relation from $SP_{\text{fin}}(\mathcal{B})$ is invariant under this operation. By results of [8, 9, 23], $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{B}))$ is tractable. Hence the algebra $\mathcal{B}$ is tractable.

Case 2. $\varphi(0) \neq \varphi(1)$.

It follows that $\varphi(2) \in \{\varphi(0), \varphi(1)\}$. Suppose that $\varphi(2) = \varphi(0)$. We have

$$\varphi(0) = \varphi(0 \circ 0) = \varphi(0) * \varphi(0) = \varphi(0) * \varphi(2) = \varphi(0 \circ 2) = \varphi(1).$$

[4] We shall write $x \circ y$ instead of $\circ(x, y)$.

If $\varphi(2) = \varphi(1)$ then

$$\varphi(0) = \varphi(1 \circ 0) = \varphi(1) * \varphi(0) = \varphi(2) * \varphi(0) = \varphi(2 \circ 0) = \varphi(1).$$

Hence this second case is impossible.

The following corollary from Theorems 3.9 and 3.10 seems to be quite significant, since all known forms of **NP**-complete **CSP**s (see [20, 23]) can be obtained using it. An algebra (B, G) where G is a permutation group on B is said to be a *G-set*. A G-set is non-trivial if its universe is not a singleton.

Corollary 3.13 *A finite algebra $\mathcal{A}$ is* **NP***-complete if it has a factor $\mathcal{B}$ term equivalent to a non-trivial G-set.*

Proof. Denote by $\mathcal{A}'$ a subalgebra of $\mathcal{A}$ such that $\mathcal{B}$ is a homomorphic image of $\mathcal{A}'$, and let B be the universe of $\mathcal{B}$.

First assume that B contains only two elements, say 0 and 1. Then the relation N (see Example 2.6) belongs to $SP_{\text{fin}}(\mathcal{B}) = Inv(G)$. It follows from the proof of Theorem 3.10 that $\mathbf{CSP}(\{N\})$ is reducible in polynomial time to $\mathbf{CSP}(\Gamma')$ for some finite $\Gamma' \subseteq SP_{\text{fin}}(\mathcal{A}')$. Since Not-All-Equal Satisfiability which corresponds to $\mathbf{CSP}(\{N\})$ is **NP**-complete [23], we see that $\mathcal{A}'$ is also **NP**-complete. Applying Theorem 3.9 we conclude that $\mathcal{A}$ is **NP**-complete.

Now suppose that $|B| \geq 3$. Then the disequality relation $\neq_B$ belongs to $SP_{\text{fin}}(\mathcal{B})$. Proceeding as in the previous case one can show that Graph $|B|$-Colorability (see Example 2.7), which is **NP**-complete [4, 19], is reducible in polynomial time to $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$. Hence $\mathcal{A}$ is **NP**-complete. □

Most of the results in this section concern local tractability of finite algebras. However, in all known cases, locally tractable algebras are also globally tractable. The authors conjecture that this is true in general.

Problem 3.14 *Prove that local tractability of a finite algebra ensures its global tractability.*

4 Strictly simple surjective algebras

Using the results of the previous section, we now consider certain special algebras. all of whose smaller factors are tractable.

Definition 4.1 *A finite algebra is called* strictly simple[5] *if all of its smaller factors are one-element, i.e. if it is simple and all of its proper subalgebras are singletons.*

[5] In some papers appearing before 1990 such algebras are called *plain*.

All finite strictly simple surjective algebras have been described by Á. Szendrei [25]. Our goal in this Section is to classify all finite strictly simple surjective algebras with respect to tractability.

In order to obtain results about global tractability, we shall consider all (not only idempotent) strictly simple surjective algebras. Note that the full idempotent reduct may not inherit the property of being strictly simple.

Our results show that, in this special case, local and global forms of tractability coincide, which confirms our conjecture in the preceding section.

To formulate Á.Szendrei's result we need to give some notation.

Let G be a permutation group on a set A. By $\mathcal{R}(G)$ we denote the clone of operations on A preserving each relation of the form $\{(a, g(a)) \mid a \in A\}$ where $g \in G$, and $\mathcal{F}(G)$ denotes the set of idempotent members of $\mathcal{R}(G)$.

A permutation group G on a set A is called *regular* if, for any $a, b \in A$, there exists $g \in G$ such that $g(a) = b$, and if each nonidentity member of G has no fixed point. G is called *primitive* if any smaller homomorphic image of the algebra (A, G) is one-element.

Let ${}_F\overline{A} = (A; +, F)$ be a finite vector space over a finite field F, $T(\overline{A})$ the group of translations $\{x + a \mid a \in A\}$, End ${}_F\overline{A}$ the endomorphism ring of ${}_F\overline{A}$. Then one can consider $\overline{A}$ as a module over End ${}_F\overline{A}$. This module is denoted by ${}_{(\mathrm{End}\ {}_F\overline{A})}\overline{A}$.

Finally, let $\mathcal{F}_k^0$ denote the clone of all operations preserving the relation

$$X_k^0 = \{(a_1, \ldots, a_k) \in A^k \mid a_i = 0 \text{ for at least one } i,\ 1 \le i \le k\}$$

where 0 is some fixed element of A, and let $\mathcal{F}_\omega^0 = \bigcap_{k=2}^{\infty} \mathcal{F}_k^0$.

Theorem 4.2 ([25]) *Let $\mathcal{A}$ be a finite strictly simple surjective algebra.*

- *If $\mathcal{A}$ has no one-element subalgebras then $\mathcal{A}$ is term equivalent to one of the following algebras:*

  (a) $(A, \mathcal{R}(G))$ for a regular permutation group G on A;

  (b) $(A, \mathrm{Clo}_{\mathrm{id}}({}_{(\mathrm{End}\ {}_K\overline{A})}\overline{A}) \cup T(\overline{A}))$ for some vector space ${}_K\overline{A} = (A; +, K)$ over a finite field K;

  (c) (A, G) for a primitive permutation group G on A.

- *If $\mathcal{A}$ has one-element subalgebras, then $\mathcal{A}$ is idempotent and term equivalent to one of the following algebras:*

  (a°) $(A, \mathcal{F}(G))$ for a permutation group G on A such that every nonidentity member of G has at most one fixed point;

  (b°) $(A, \mathrm{Clo}_{\mathrm{id}}({}_{(\mathrm{End}\ {}_K\overline{A})}\overline{A}))$ for some vector space ${}_K\overline{A}$ over a finite field K;

  (d) $(A, \mathcal{F}(G) \cap \mathcal{F}_k^0)$ for some k $(2 \le k \le \omega)$, some element $0 \in A$ and some permutation group G on A such that 0 is the unique fixed point of every nonidentity member of G;

  (e) (A, F) where $|A| = 2$ and F contains a semilattice operation;

  (f) a two-element algebra with empty set of basic operations.

In the following theorem we determine all tractable finite strictly simple surjective algebras by analyzing the cases given in Theorem 4.2.

Theorem 4.3 *A finite strictly simple surjective algebra $\mathcal{A}$ is* **NP***-complete if it is term equivalent to (A, G) for a primitive permutation group G on A, or to a two-element algebra with an empty set of basic operations. Otherwise $\mathcal{A}$ is tractable.*

Proof. Applying Corollary 3.13, we see that algebras in cases (c) and (f) are **NP**-complete.

Let $\mathcal{A}$ be an algebra of type (a) or (a°). We prove that the dual discriminator operation $d(x, y, z)$ is a term operation of $\mathcal{A}$. It is easy to verify that d preserves every relation of the form $\{(a, g(a)) \mid a \in A\}$ where g is a permutation on A. Since d is an idempotent operation, it belongs to both $\mathcal{R}(G)$ and $\mathcal{F}(G)$. Hence, any relation in $SP_{\mathrm{fin}}(\mathcal{A})$ is invariant under d. It follows from [9] that $\mathbf{CSP}(SP_{\mathrm{fin}}(\mathcal{A}))$ is tractable. So, $\mathcal{A}$ is tractable.

If $\mathcal{A}$ is an algebra of type (b) or (b°) then the affine operation $f(x, y, z) = x - y + z$ is a term operation of $\mathcal{A}$. Tractability of $\mathcal{A}$ follows from results of [3, 9].

Now let $\mathcal{A}$ be an algebra of type (d) corresponding to some $2 \leq k \leq \omega$. Let $l = \min\{k, |A|\}$. Consider an operation $f_l(x_1, \dots, x_{l+1})$ defined as follows:

$$f_l(x_1, \dots, x_{l+1}) = \begin{cases} a, & \text{if } |\{i : x_i = a\}| \geq l, \\ 0 & \text{otherwise.} \end{cases}$$

We now verify that $f_l \in \mathcal{F}(G) \cap \mathcal{F}^0_k$. First we prove that f_l preserves any relation of the form $g^\circ = \{(a, g(a)) \mid a \in A\}$ where $g \in G$. Let M be a $2 \times (l+1)$ matix whose columns are pairs from g°. If some l elements in the first row of M are equal to some $a \in A$ then the corresponding elements in the second row coincide with $g(a)$, and, by definition of f_l, we have that the pair $(a, g(a))$ corresponding to $f(M)$ belongs to g°. If no l elements of the first row coincide then, since g is a permutation, the same holds for the second row, and the pair $(0, 0)$ corresponding to $f(M)$ also belongs to g° because 0 is a fixed point of g. Hence $f_l \in \mathcal{F}(G)$.

Now we check that $f_l \in \mathcal{F}^0_k$. Let M be a $k \times (l+1)$ matrix whose columns all belong X^0_k, that is each column contains 0. By the choice of l, there exist a row of M containinig two entries of 0. By the definition of f_l, the value of this operation on the row must be equal to 0. We see that $f(M)$ belongs to X^0_k. Therefore $f_l \in \mathcal{F}(G) \cap \mathcal{F}^0_k$.

Hence, any relation in $SP_{\mathrm{fin}}(\mathcal{A})$ is invariant under f_l.

It is easy to see that f_l is a near-unaminity operation. It was proved in [10] that in this case $\mathbf{CSP}(SP_{\mathrm{fin}}(\mathcal{A}))$ is tractable. So, $\mathcal{A}$ is tractable.

In case (e) tractability of $\mathcal{A}$ follows from [23]. □

We see that, as in Theorem 2.24, any finite strictly simple surjective algebra is either tractable or **NP**-complete. Whether there is such a dichotomy in general is possibly the most intriguing open question in this area.

Problem 4.4 *Is every finite algebra either tractable or* **NP***-complete?*

The results obtained via the algebraic approach and, in particular, the results obtained in this paper, prompt us to make the following conjecture.

Conjecture 4.5 *A finite algebra is tractable if and only if it has no factor which is term equivalent to a nontrivial G-set.*

5 Conclusion

In this paper we have shown how the theory of universal algebras can be used in studying the structure of constraint satisfaction problems.

We have introduced the notion of a tractable algebra and established some general links between the tractability of an algebra and the tractability of its derived algebras. We have also significantly reduced the types of algebras which need to be studied.

We believe that this framework will be very useful in the further analysis of constraint satisfaction problems.

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